D2.5.6 Fuzzy Reasoning Extensions

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Today more and more applications from different research domains are using Semantic Web languages, like RDF and OWL, in order to build knowledge based systems. Unfortunately, many of these applications are facing a vast amount of imprecise and vague knowledge and information. To overcome these limitations fuzzy extensions to Semantic Web languages have been proposed. The current deliverable presents some very recent results on fuzzy extensions to ontology languages. This work builds upon previous results reported in deliverables 2.5.1, 2.5.2, and 2.5.3, which presented the syntax, semantics and reasoning capabilities for fuzzy extensions of ontology and rule languages.

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Executive Summary

Today more and more applications from different research domains are using Semantic Web languages, like RDF and OWL, in order to build knowledge based systems. Unfortunately, many of these applications are facing a vast amount of imprecise and vague knowledge and information. To overcome these limitations fuzzy extensions to Semantic Web languages have been proposed. The current deliverable presents some very recent results on fuzzy extensions to ontology languages. This work builds upon previous results reported in deliverables 2.5.1, 2.5.2, and 2.5.3, which presented the syntax, semantics and reasoning capabilities for fuzzy extensions of ontology and rule languages.

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Chapter 1

Introduction

The management of uncertainty has become a very important issue in knowledge representation and reasoning during the last decade. There are several applications within different research domains that face a huge amount of uncertain and imprecise information that is very important to be captured and dealt with. This is especially the case in (semi)automatic procedures, like robotics [KKC94], decision making [Zim87], multimedia processing [SST$^+$05b], medical diagnosis [GBDG05], and many more. For that reason there are several works that extend logical formalisms with uncertainty and imprecision handling frameworks. One logical formalism that has gained considerable attention the last decade is Description Logics (DLs) [BN03]. Description Logics are a family of class-based (concept-based) knowledge representation formalisms, equipped with well-defined model-theoretic semantics.

Although there exist DL languages with considerable expressive power, like the $\mathcal{SHIN}$ language, they feature expressiveness limitations regarding their ability to represent vague and imprecise knowledge. Consider for example the case of the identification of brain anatomical structures in MRI (Magnetic Resonance Images) images. In such an application the goal is to (semi)automatically segment and identify the various parts of the brain by providing appropriate labels for each of these parts. Such a process can be assisted by a brain cortex anatomy knowledge base, which formally describes the various parts of the brain anatomy [GBDG05]. For example, we could have the entities,

$$\begin{aligned} \mathsf{OPIFGyrus} &\sqsubseteq \exists \mathsf{isDAPartOf}.\mathsf{IFGyrus} \\ \mathsf{IFGyrus} &\sqsubseteq \exists \mathsf{isDAPartOf}.\mathsf{FrontalLobe} \end{aligned}$$

where $\sqsubseteq$ is a subsumption (implication) relation, $\mathsf{OPIFGyrus}$ represents the Orbital Pars of Interior Frontal Gyrus, $\mathsf{IFGyrus}$ the Inferior Frontal Gyrus [DGM04] and $\mathsf{isDAPartOf}$ represents the relation, $\mathsf{isDirectAnatomicalPartOf}$. Furthermore, using $\mathcal{SHIN}$ one can capture the fact that, $\mathsf{isDAPartOf}$ is a sub-relation of a more broader relation, $\mathsf{isAPartOf}$, that the relation $\mathsf{hasDAPart}$ is an inverse of $\mathsf{isDAPartOf}$ and that $\mathsf{isAPartOf}$ is a transitive relation. Following [GBDG05], we can specify that $\mathsf{hasDAPart}$ is an *inverse-functional* relation, writing $\top \sqsubseteq \leq 1\,\mathsf{Inv}(\mathsf{hasDAPart})$ and meaning that there can be at-most one object that has some other object as a direct anatomical part. Now suppose that an image segmentation algorithm is applied to an MRI image in order to identify different brain parts. Since such algorithms cannot be sure about the membership or non-membership of an object to a certain concept, they usually provide confidence (truth) degrees. For example, we could have that o_1 $\mathsf{isDAPartOf}$ o_2 to a degree of 0.8, o_2 $\mathsf{isDAPartOf}$ o_3 to a degree of 0.9, o_2 $\mathsf{isDAPartOf}$ o_3' to a degree of 0.3, o_1 belongs to $\mathsf{OPIFGyrus}$ to a degree of 0.75, o_2 belongs to $\mathsf{IFGyrus}$ to a degree of 0.85 and that o_2 belongs to $\leq 1\,\mathsf{Inv}(\mathsf{hasDAPart})$ to a degree of 0.7, meaning that it is likely that o_2 is connected only with one individual. From that fuzzy knowledge one could deduce that o_3 belongs to $\exists\mathsf{hasAPart}.\mathsf{OPIFGyrus}$ to a degree of 0.75. In order to make applications that use DLs able to cope with such information we have to extend them with a theory capable of representing such kind of information as well as to provide practical reasoning algorithms. One such theory is fuzzy set theory.

In previous deliverables of this work package, namely 2.5.1, 2.5.2 and 2.5.3, we have provided some first results towards extending the syntax and semantics of expressive Semantic Web languages with fuzzy set theory. More precisely, in deliverable D2.5.1 we have presented some first results about extending the OWL and the SWRL languages with fuzzy set theory. In deliverable D2.5.2 we have investigated the problem of querying fuzzy knowledge bases, while in deliverable D2.5.3 we have presented the syntax, semantics and a reasoning algorithm for the fuzzy DL language f_{KD}-$\mathcal{SI}$. These very first results have since been extended leading to very impressive and mature results, like reasoning algorithms for expressive fuzzy DLs, a prototype reasoner that implements such reasoning algorithms, decision procedures for handling GCIs in the context of fuzzy DLs, syntax and semantics of OWL and semantics of fuzzy extensions to Semantic Web rule languages, like SWRL and RuleML. In the current deliverable we will report on this progress.

1.1 Reader's Guide

The rest of the deliverable is organized as follows. In Chapter 2 we provide a short introduction to Fuzzy Set theory, which is necessary for the rest of the Deliverable. Chapter 3 presents the syntax and semantics of the fuzzy $\mathcal{SHOIN}$ language. We further present several properties of the semantics of the extended language that differ from the properties of the classical (crisp) $\mathcal{SHOIN}$ DL and we provide a reasoning algorithm for the very expressive fuzzy DL language f_{KD}-$\mathcal{SHOIN}$. Furthermore, we present a prototype implementation of a reasoning engine for the fuzzy DL langauge f_{KD}-$\mathcal{SHIN}$. In Chapter 4, we present the syntax and semantics of a fuzzy extension to the OWL language. We also provide the extended abstract and concrete syntax of fuzzy OWL, and present a reduction algorithm that reduces the key inference problems of fuzzy OWL to the inference problems of fuzzy DLs. Additionally, in Chapter 5 we present a fuzzy extension to several rule languages, focusing on the SWRL language and the RuleML markup language. Finally, Chapter 6 concludes the deliverable by summarizing our results on fuzzy reasoning extensions, and by presenting a number of open issues and future research directions we are investigating regarding such extensions.

Chapter 2

Fuzzy Set Preliminaries

Fuzzy set theory and fuzzy logic are widely used for capturing imprecise and vague knowledge [KY95]. While in classical set theory an element either belongs to a set or not, in fuzzy set theory elements belong only to a certain degree. More formally, let *X* be a collection of elements (the universe of discourse), i.e. $X = \{x_1, x_2, \ldots\}$. A crisp subset *S* of *X* is any collection of elements of *X* that can be defined with the aid of its *characteristic function* χ_S*(x)* that assigns any $x \in X$ to a value 1 or 0 if this element belongs to *X* or not, respectively. Hence, χ_S*(x)* is a function of the form

$$\chi_S(x) : X \to 0, 1.$$

On the other hand, a fuzzy subset *A* of *X*, is defined by a *membership function* μ_A*(x)*, or simply *A(x)*, $x \in X$. This membership function assigns any $x \in X$ to a value between 0 and 1 that represents the degree in which this element belongs to *X*. Similarly, we can define fuzzy relations. A fuzzy relation R over *X* × *X* is defined by a function which, given a pair of elements $\langle x, y \rangle$ returns the degree that the pair belongs to the fuzzy relation. Furthermore, the most important operations and properties defined on crisp sets and relations, like complement, union, intersection, transitivity etc, are extended in order to cover fuzzy sets and fuzzy relations, thus creating a sound mathematical theory which is today applied successfully in many applications.

2.1 Fuzzy Set Theoretic Operations

In this section, we will explain how to extend boolean operations and logical implications in fuzzy sets and fuzzy logics. In the current framework these operations are performed by mathematical functions over the unit interval. These functions are usually called *norm* operations [KY95], and satisfy specific properties.

The operation of a fuzzy complement (c) is a unary operation, defined by a function of the form $c : [0,1] \to [0,1]$. In order to produce meaningful fuzzy complements, these functions must satisfy certain properties. More precisely, they must satisfy the *boundary conditions*, $c(0) = 1$ and $c(1) = 0$, and must be *monotonic decreasing*, for $a \leq b$, $c(a) \geq c(b)$. Additionally, here, we are interested in complements that are also *continuous* and *involutive*, for each $a \in [0,1]$ $c(c(a)) = a$, holds. Many widely used fuzzy complements, like the Lukasiewicz negation, $c_L(a) = 1 - a$, the Sugeno class, $c_S(a) = \frac{1-a}{1+\lambda a}$, $\lambda \in (-1,\infty)$ and the Yager class, $c_Y(a) = (1 - a^w)^{1/w}$, $w \in (0,\infty)$, of fuzzy complements satisfy these properties. One non-involutive fuzzy complement is the Gödel complement given by, $c(a) = 0$ if $a > 0$, otherwise $c(0) = 1$.

The operation of fuzzy intersection is performed by a function of the form $t : [0,1] \times [0,1] \to [0,1]$, called *t-norm* [KY95] operation. These functions satisfy the *boundary condition*, $t(a,1) = a$, are *monotonic increasing*, for $b \leq d$ then $t(a,b) \leq t(a,d)$, *commutative*, $t(a,b) = t(b,a)$ and *associative*, $t(a,t(b,c)) = t(t(a,b),c)$. Usually, t-norm operations are also considered to be *continuous* and *subidempotent*, $t(a,a) < a$, for $a \in (0,1)$. Such norms are called *Archimedean* t-norms. The only *idempotent* t-norm is the Gödel t-norm given by, $t_G(a,b) = \min(a,b)$. It can be proved that for any t-norm t it holds that, $a,b \geq t(a,b)$, and $t(a,0) = 0$. Commonly used Archimedean t-norms are the Lukasiewicz t-norm $t_L(a,b) = \max(0, a+b-1)$, and the product t-norm $t_P(a,b) = a \cdot b$.

The operation of fuzzy union is performed by a function $u : [0,1] \times [0,1] \to [0,1]$, called *t-conorm*. Similarly to t-norms, these functions satisfy the *boundary condition*, $u(a,0) = a$, are monotonic increasing, commutative and associative. In many cases t-conorms are continuous and *superidempotent*, $u(a,a) > a$, for $a \in (0,1)$. Such norms are called *Archimedean* t-conorms. The only idempotent t-conorm is the Gödel t-conorm given by, $u_G(a,b) = \max(a,b)$. It can be proved that for any t-conorm u it holds that, $a,b \leq u(a,b)$, and $u(a,1) = 1$. Commonly used Archimedean t-conorms are the Lukasiewicz t-conorm $u_L(a,b) = \min(1, a+b)$, and the probabilistic sum $u_P(a,b) = a + b - a \cdot b$.

The operation of fuzzy implication is performed by a function of the form $\mathcal{J} : [0,1] \times [0,1] \to [0,1]$. Two distinct classes of fuzzy implications are commonly used in fuzzy logic. The first one results from the extension of the proposition $\neg a \vee b$ with fuzzy operators. Thus, we get the class of S-implications, which are defined by the operation $\mathcal{J}(a,b) = u(c(a),b)$ [KY95]. The second class of fuzzy implications results from the proposition $\max\{x \in [0,1] \mid a \wedge x \leq b\}$, which is an alternative expression for logical implication. Thus, we get the class of R-implications, which are given by the equation $\mathcal{J}(a,b) = \sup\{x \in [0,1] \mid t(a,x) \leq b\}$ [KY95]. This operation is usually referred to as ω_t operation. For all R-implications $\omega_t(a,b) = 1$ iff $a \leq b$ [Haj98]. Commonly used R-implications are the Lukasiewicz implication $\mathcal{J}_L(a,b) = \min(1, 1 - a + b)$, the Gödel implication, $\mathcal{J}_G(a,b) = b$, if $a > b$, and the Goguen implication, $\mathcal{J}_P(a,b) = a/b$, if $a > b$, while for S-implications the Kleene-Dienes implication, $\mathcal{J}_{KD}(a,b) = \max(1 - a, b)$.

We conclude that in order to define a fuzzy logic we need to specify the fuzzy operations, c, t, u and $\mathcal{J}$, that we are going to use. Such a collection of operations would be referred to as a *fuzzy quadruple*, $\langle c, t, u, \mathcal{J} \rangle$, or *fuzzy triple* in the case of $\langle c, t, u \rangle$.

Chapter 3

Fuzzy Description Logics

In this chapter we will provide a full account of the fuzzy DL f-$\mathcal{SHOIN}$. More precisely, we will present the syntax, semantics and the inference problems of the f-$\mathcal{SHOIN}$ language. Regarding the latter we will emphasize on how to deal with General Concept Inclusion axioms in the context of fuzzy DLs. Finally, we will present a tableaux reasoning algorithm to decide the key inference problems of f$_{KD}$-$\mathcal{SHOIN}$ (see next section for a definition of f$_{KD}$-$\mathcal{SHOIN}$) and we will present some initial results on a prototype implementation of the presented reasoning algorithm.

The syntax and semantics of f-$\mathcal{SHOIN}$ were first presented in [Str05], but our presentation here differs from [Str05] in various points, like the semantics of concept and role inclusion axioms, the semantics of number restrictions which are based on the extensions presented in [SSP06] and the semantics of nominals. Moreover, note that the reasoning algorithm for the language f$_{KD}$-$\mathcal{SI}$ was first reported in Deliverable 2.5.3 [PFT$^+$05]. Moreover, the algorithms for f$_{KD}$-$\mathcal{SHIN}$, f$_{KD}$-$\mathcal{SHOIN}$ and the investigation of GCIs have been presented in [SST$^+$05a], [SST$^+$05c] and [SSSP06], respectively.

3.1 Syntax and Semantics of fuzzy $\mathcal{SHOIN}$

As usual we have an alphabet of distinct concept names ($\mathbf{C}$), role names ($\mathbf{R}$) and individual names ($\mathbf{I}$). f-$\mathcal{SHOIN}$-roles and f-$\mathcal{SHOIN}$-concepts are defined as follows:

Definition 1 *Let $RN \in \mathbf{R}$ be a role name and R an f-$\mathcal{SHOIN}$-role. f-$\mathcal{SHOIN}$-roles are defined by the abstract syntax: $R ::= RN \mid R^-$, where R^- denotes the inverse of the role R. The inverse relation of roles is symmetric, and to avoid considering roles such as R^{--}, we define a function* Inv *which returns the inverse of a role, more precisely* $\mathsf{Inv}(RN) := RN^-$ *and* $\mathsf{Inv}(RN^-) := RN$*. The set of f-$\mathcal{SHOIN}$-concepts is the smallest set such that*

1. *every concept name $CN \in \mathbf{C}$ is an f-$\mathcal{SHOIN}$-concept,*

2. *if* $o \in \mathbf{I}$ *then* $\{o\}$ *is an f-*$\mathcal{SHOIN}$*-concept,*

3. *if* C *and* D *are f-*$\mathcal{SHOIN}$*-concepts,* R *an f-*$\mathcal{SHOIN}$*-role,* S *a* simple *f-*$\mathcal{SHOIN}$*-role* [1] *and* $p \in \mathbb{N}$*, then* $(C \sqcup D)$*,* $(C \sqcap D)$*,* $(\neg C)$*,* $(\forall R.C)$*,* $(\exists R.C)$*,* $(\geq pS)$ *and* $(\leq pS)$ *are also f-*$\mathcal{SHOIN}$*-concepts.*

By allowing p to take only the values 0 and 1, i.e. concepts of the form $\leq 1R$, $\geq 1R$ and $\leq 0R$, and by removing point 2 in the above definition we obtain the set of f-$\mathcal{SHIF}$-concepts. As we can see, f-$\mathcal{SHOIN}$-concepts are formed by the same abstract syntax as that of crisp $\mathcal{SHOIN}$-concepts [HS05]. Ont the other hand the semantics of fuzzy DLs are provided by a *fuzzy interpretation* [Str01]. A fuzzy interpretation is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where the domain $\Delta^{\mathcal{I}}$ is a non-empty set of objects and $\cdot^{\mathcal{I}}$ is a *fuzzy interpretation function*, which maps

1. an individual name $a \in \mathbf{I}$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$,

2. a concept name $\mathsf{A} \in \mathbf{C}$ to a membership function $\mathsf{A}^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0, 1]$,

3. a role name $R \in \mathbf{R}$ to a membership function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0, 1]$.

Intuitively, an object (pair of objects) can now belong to a fuzzy concept (role) to any degree between 0 and 1. For example, $\mathsf{HotPlace}^{\mathcal{I}}(\mathtt{Rome}^{\mathcal{I}}) = 0.7$, means that $\mathtt{Rome}^{\mathcal{I}}$ is a hot place to a degree equal to 0.7. Moreover, fuzzy interpretations can be extended to interpret f-$\mathcal{SHOIN}$-concepts and roles, with the aid of the fuzzy set theoretic operations, defined in section 2.1. For example, since $C \sqcup D$ represents a union, then by using a fuzzy union (u) $(C \sqcup D)^{\mathcal{I}}$ can be interpreted as $u(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$. Furthermore, since a value restriction $\forall R.C$ is an implication of the form, $\forall y(R(x, y) \to C(y))$, we can interpret $\forall$ as infimum $\inf$, and $\to$ as a fuzzy implication and have the equation, $\inf_{b \in \Delta^{\mathcal{I}}}\{\mathcal{J}(R^{\mathcal{I}}(a, b), C^{\mathcal{I}}(b))\}$. The complete semantics are depicted in Table 3.1, where $\sup$ represent the supremum of a set and $\inf$ the infimum.

A fuzzy $TBox$ is a finite set of fuzzy concept axioms. Let C and D be f-$\mathcal{SHOIN}$-concepts. Fuzzy concept axioms of the form $C \sqsubseteq D$ are called *fuzzy inclusion axioms*, while fuzzy concept axioms of the form $C \equiv D$ are called *fuzzy equivalence axioms*. Axioms of the form $C \sqsubseteq D$ where C is a concept description, are called *General Concept Inclusions* (GCIs). A fuzzy interpretation $\mathcal{I}$ satisfies $C \sqsubseteq D$ if $\forall a \in \Delta^{\mathcal{I}}$, $C^{\mathcal{I}}(a) \leq D^{\mathcal{I}}(a)$ and it satisfies $C \equiv D$ if $C^{\mathcal{I}}(a) = D^{\mathcal{I}}(a)$. Finally, a fuzzy interpretation $\mathcal{I}$ satisfies an f-$\mathcal{SHOIN}$ TBox $\mathcal{T}$ if it satisfies each axiom in $\mathcal{T}$; then we say that $\mathcal{I}$ is a *model* of $\mathcal{T}$. Please note that we give a crisp subsumption of fuzzy concepts. This is the usual way subsumption is defined in the context of fuzzy DLs [Str01, ST04, HKS02, SST^{+}05b, SSSP06] and fuzzy sets [KY95]. These semantics differ from the ones in [Str05], where a *fuzzy*

[1] A role is called *simple* if it is neither transitive nor has any transitive sub-roles. Allowing only simple roles to participate in number restrictions is crucial in order to get a decidable logic [HST99].

Table 3.1: Syntax and Semantics of f-$\mathcal{SHOIN}$-concepts

Constructor	DL Syntax	Semantics
top concept	$\top$	$\top^{\mathcal{I}}(a) = 1$
bottom	$\bot$	$\bot^{\mathcal{I}}(a) = 0$
conjunction	$C \sqcap D$	$(C \sqcap D)^{\mathcal{I}}(a) = t(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$
disjunction	$C \sqcup D$	$(C \sqcup D)^{\mathcal{I}}(a) = u(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$
negation	$\neg C$	$(\neg C)^{\mathcal{I}}(a) = c(C^{\mathcal{I}}(a))$
nominal	$\{o\}$	$\{o\}^{\mathcal{I}}(a) = 1$ if $a \in \{o^{\mathcal{I}}\}$, otherwise $\{o\}^{\mathcal{I}}(a) = 0$
existential restriction	$\exists R.C$	$(\exists R.C)^{\mathcal{I}}(a) = \sup_{b \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b))$
value restriction	$\forall R.C$	$(\forall R.C)^{\mathcal{I}}(a) = \inf_{b \in \Delta^{\mathcal{I}}} \mathcal{J}(R^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b))$
at-least restriction	$\geq pR$	$(\geq pR)^{\mathcal{I}}(a) = \sup_{b_1,\ldots,b_p \in \Delta^{\mathcal{I}}} t(t_{i=1}^{p} R^{\mathcal{I}}(a,b_i), t_{i<j}\{b_i \neq b_j\})$
at-most restriction	$\leq pR$	$(\leq pR)^{\mathcal{I}}(a) = \inf_{b_1,\ldots,b_{p+1} \in \Delta^{\mathcal{I}}} \mathcal{J}(t_{i=1}^{p+1} R^{\mathcal{I}}(a,b_i), u_{i<j}\{b_i = b_j\})$
inverse role	R^-	$(R^-)^{\mathcal{I}}(b,a) = R^{\mathcal{I}}(a,b)$
datatype exists restriction	$\exists T.d$	$(\exists T.d)^{\mathcal{I}}(a) = \sup_{y \in \Delta_{\mathbf{D}}} T(s^{\mathcal{I}}(a,y), y \in d^{\mathcal{I}})$
datatype value restriction	$\forall T.d$	$(\forall T.d)^{\mathcal{I}}(a) = \inf_{y \in \Delta_{\mathbf{D}}} \mathcal{J}(T^{\mathcal{I}}(a,y), y \in d^{\mathcal{I}})$
datatype at-least	$\geq pT$	$(\geq pT)^{\mathcal{I}}(a) = \sup_{y_1,\ldots,y_p \in \Delta_{\mathbf{D}}} t(t_{i=1}^{p} R^{\mathcal{I}}(a,y_i), t_{i<j}\{y_i \neq y_j\})$
datatype at-most	$\leq pT$	$(\leq pT)^{\mathcal{I}}(a) = \inf_{y_1,\ldots,y_{p+1} \in \Delta_{\mathbf{D}}} \mathcal{J}(t_{i=1}^{p+1} R^{\mathcal{I}}(a,y_i), u_{i<j}\{y_i = y_j\})$

subsumption of fuzzy concepts was provided. Syntactically, fuzzy subsumption is defined by an axiom of the form $\langle C \sqsubseteq D, n\rangle$, where $n \in [0,1]$, while a fuzzy interpretation $\mathcal{I}$ satisfies such an axiom if $\inf_{x \in \Delta^{\mathcal{I}}} \mathcal{J}(C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)) \geq n$ [Str05]. We choose not to introduce such axioms in the current approach since they will impose many syntactic changes in the f-OWL language which are difficult to encode and implement.

A fuzzy $RBox$ is a finite set of fuzzy role axioms. Fuzzy role axioms of the form $\mathsf{Trans}(R)$, are called *fuzzy transitive role* axioms; fuzzy role axioms of the form $R \sqsubseteq S$ are called *fuzzy role inclusion* axioms. A fuzzy interpretation $\mathcal{I}$ satisfies an axiom $\mathsf{Trans}(R)$ if $\forall a, c \in \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(a,c) \geq \sup_{b \in \Delta^{\mathcal{I}}}\{t(R^{\mathcal{I}}(a,b), R^{\mathcal{I}}(b,c))\}$ and it satisfies $R \sqsubseteq S$ if $\forall \langle a,b\rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(a,b) \leq S^{\mathcal{I}}(a,b)$. Finally, $\mathcal{I}$ satisfies an f-$\mathcal{SHOIN}$ RBox if it satisfies each role axiom in $\mathcal{R}$; in this case we say that $\mathcal{I}$ is a model of $\mathcal{R}$.

A fuzzy $ABox$ is a finite set of fuzzy assertions. A *fuzzy assertion* [Str01] is of the form $(a : C) \bowtie n$, $(\langle a,b\rangle : R) \bowtie n$, where $\bowtie \in \{\geq, >, \leq, <\}$, or of the form $a \not\doteq b$, for $a, b \in \mathbf{I}$. In many cases we write $(a : C) = n$ instead of writing two fuzzy assertions of the form $(a : C) \geq n$ and $(a : C) \leq n$. We call assertions containing either $\geq$ or $>$ *positive* assertions, while those containing either $\leq$ or $<$ *negative* assertions. A fuzzy interpretation $\mathcal{I}$ satisfies $(a : C) \geq n$ if $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n$, it satisfies $(\langle a,b\rangle : R) \geq n$ if $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n$. An assertion of the form $a \not\doteq b$ is satisfied if $a^{\mathcal{I}} \neq b^{\mathcal{I}}$ and the satisfiability of fuzzy assertions with $>, \leq$ and $<$ is defined analogously. A fuzzy interpretation $\mathcal{I}$ satisfies a fuzzy $ABox$ $\mathcal{A}$ if it satisfies all fuzzy assertions in $\mathcal{A}$. In this case, we say that $\mathcal{I}$ is a model of $\mathcal{A}$. If $\mathcal{A}$ has a model then we say that it is *consistent*.

As noted in [SSP06], since we have defined fuzzy number restrictions it is possible that $(a : (\geq p_1 R.C)) \geq n_1$ and $(a : (\leq p_2 R.C)) \geq n_2$, with $p_1 > p_2$ and $n_1, n_2 \in [0, 1]$ simultaneously hold, without forming a contradiction. More precisely if t is the Gödel t-norm and $\mathcal{J}$ the KD-implication we have,

Lemma 2 *Let $\mathcal{A} = \{(a : (\geq p_1 R.C)) \geq n_1, (a : (\leq p_2 R.C)) \geq n_2\}$ be a fuzzy ABox, with $n_1, n_2 \in [0, 1]$, $p_1, p_2 \in \mathbb{N}$, and $p_2 < p_1$. Then $\mathcal{A}$ is satisfiable iff $n_1 + n_2 \leq 1$.*

In classical DLs, since $n_1, n_2 \in \{0, 1\}$, the inequality $n_1 + n_2 \leq 1$ is satisfied if and only if either $n_1 = 0$ or $n_2 = 0$. Indeed in crisp DLs an individual cannot simultaneously belong to both such concepts.

Now we introduce some notation. In the following we use the symbols $\rhd$ and $\lhd$ as a placeholder for the inequalities $\geq, >$ and $\leq, <$, respectively. Additionally, we use the symbols $\rhd^-, \lhd^-$ to denote the *reflection* of an inequality. For example the reflection of $\geq$ is $\leq$, while the reflection of $<$ is $>$. Furthermore, we use the symbol $+$ to denote the *strengthening* or *weakening* of an inequality. For example applying $+$ to $\geq$ gives $>$, i.e. strengthens the inequality, while applying it to $>$ gives $\geq$, i.e. weakens the inequality. Finally, by $\neg\bowtie$ to denote the *negation* of an inequality. For example, the negation of $\geq$ is $<$ and that of $<$ is $\geq$.

A *fuzzy knowledge base* Σ is a triple $\langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$, that contains a fuzzy $TBox$, $RBox$ and $ABox$, respectively.

Example 1 *Consider the fuzzy knowledge base (Σ), we introduced in Chapter 1. Formally written such a fuzzy knowledge base can be defined as follows:*

$$
\begin{aligned}
\mathcal{T} &= \{\mathsf{OPIFGyrus} \sqsubseteq \exists \mathsf{isDAPartOf}.\mathsf{IFGyrus}, \\
&\quad\ \mathsf{IFGyrus} \sqsubseteq \exists \mathsf{isDAPartOf}.\mathsf{FrontalLobe}\}, \\
\mathcal{A} &= \{(\langle \mathsf{o}_1, \mathsf{o}_2 \rangle : \mathsf{isDAPartOf}) \geq 0.8, \\
&\quad\ (\langle \mathsf{o}_2, \mathsf{o}_3 \rangle : \mathsf{isDAPartOf}) \geq 0.9, \\
&\quad\ (\langle \mathsf{o}_4, \mathsf{o}_3 \rangle : \mathsf{isDAPartOf}) \geq 0.3, \\
&\quad\ (\mathsf{o}_1 : \mathsf{OPIFGyrus}) \geq 0.75, (\mathsf{o}_2 : \mathsf{IFGyrus}) \geq 0.85, \\
\mathcal{R} &= \{\mathsf{Trans}(\mathsf{isAPartOf}), \mathsf{isDAPartOf} \sqsubseteq \mathsf{isAPartOf}, \\
&\quad\ \mathsf{hasAPart} \sqsubseteq \mathsf{isAPartOf}^-, \mathsf{isAPartOf}^- \sqsubseteq \mathsf{hasAPart}\}.
\end{aligned}
$$

The last two role inclusion axioms state the fact that $\mathsf{hasAPart}$ *is the inverse of the role* $\mathsf{isAPartOf}$*. Now let $\mathcal{I}$ be a fuzzy interpretation that is a model of the KB. This means that $\mathcal{I}$ satisfies $\mathcal{A}$, hence the following inequalities must hold,* $\mathsf{isDAPartOf}^{\mathcal{I}}(\mathsf{o}_1^{\mathcal{I}}, \mathsf{o}_2^{\mathcal{I}}) \geq 0.8$, $\mathsf{isDAPartOf}^{\mathcal{I}}(\mathsf{o}_2^{\mathcal{I}}, \mathsf{o}_3^{\mathcal{I}}) \geq 0.9$, $\mathsf{isDAPartOf}^{\mathcal{I}}(\mathsf{o}_4^{\mathcal{I}}, \mathsf{o}_3^{\mathcal{I}}) \geq 0.3$, $\mathsf{OPIFGyrus}^{\mathcal{I}}(\mathsf{o}_1^{\mathcal{I}}) \geq 0.75$, $\mathsf{IFGyrus}^{\mathcal{I}}(\mathsf{o}_2^{\mathcal{I}}) \geq 0.85$ *and* $\leq 1\, \mathsf{Inv}(\mathsf{isDAPartOf})^{\mathcal{I}}(\mathsf{o}_3^{\mathcal{I}}) \geq 0.7$.

Furthermore, $\mathcal{I}$ should also satisfy the two subsumption axioms of the TBox. Hence, we have that $\forall \mathsf{o}_i \in \Delta^{\mathcal{I}}$ both inequalities $(\exists \mathsf{isDAPartOf}.\mathsf{IFGyrus})^{\mathcal{I}}(\mathsf{o}_i^{\mathcal{I}}) \geq \mathsf{OPIFGyrus}^{\mathcal{I}}(\mathsf{o}_i^{\mathcal{I}})$

and $(\exists\mathsf{isDAPartOf.FrontalLobe})^{\mathcal{I}}(\mathsf{o}_i^{\mathcal{I}}) \geq \mathsf{IFGyrus}^{\mathcal{I}}(\mathsf{o}_i^{\mathcal{I}})$ *must hold. In order to satisfy these inequations we have to carefully define our fuzzy interpretation* $\mathcal{I}$*. Suppose that we use the t-norm* $t(a,b) = \min(a,b)$ *for performing fuzzy intersection. Then, if we consider taking the equalities, i.e.* $\mathsf{isDAPartOf}^{\mathcal{I}}(\mathsf{o}_1^{\mathcal{I}}, \mathsf{o}_2^{\mathcal{I}}) = 0.8$*,* $\mathsf{OPIFGyrus}^{\mathcal{I}}(\mathsf{o}_1^{\mathcal{I}}) = 0.75$*, similarly with the rest of concepts and roles, then the first inequality holds, while the second one does not hold since* $\mathsf{FrontalLobe}^{\mathcal{I}}(\mathsf{o}_3^{\mathcal{I}}) = 0$*. On the other hand we can define* $\mathcal{I}$ *to assign,* $\mathsf{FrontalLobe}^{\mathcal{I}}(\mathsf{o}_3^{\mathcal{I}}) \in [0.85, 1]$ *for which the second subsumption axiom is satisfied. Please note that the fuzzy interpretation* $\mathcal{I}$ *that results by considering the equalities, as well as the last restriction, is one out of infinitely many fuzzy interpretations that can be defined. For example, another interpretation* $\mathcal{I}$*' can assign different membership degrees, but in order for* $\mathcal{I}$*' to be a model of* $\mathcal{A}$ *and* $\mathcal{T}$ *the above inequalities must hold.*

In order for $\mathcal{I}$ *to be a model of the knowledge base it should also satisfy the axioms of the RBox. Hence, from the role inclusion* $\mathsf{isDAPartOf} \sqsubseteq \mathsf{isAPartOf}$ *we have that* $\mathsf{isAPartOf}^{\mathcal{I}}(\mathsf{o}_1^{\mathcal{I}}, \mathsf{o}_2^{\mathcal{I}}) \geq 0.8$*,* $\mathsf{isAPartOf}^{\mathcal{I}}(\mathsf{o}_2^{\mathcal{I}}, \mathsf{o}_3^{\mathcal{I}}) \geq 0.9$ *and* $\mathsf{isAPartOf}^{\mathcal{I}}(\mathsf{o}_4^{\mathcal{I}}, \mathsf{o}_3^{\mathcal{I}}) \geq 0.3$*; similarly for the rest of the role inclusions. Moreover, since* $\mathsf{isAPartOf}$ *is transitive and due to the semantics of inverse roles we have,* $(\mathsf{isAPartOf}^-)^{\mathcal{I}}(\mathsf{o}_3^{\mathcal{I}}, \mathsf{o}_1^{\mathcal{I}}) = \mathsf{isAPartOf}^{\mathcal{I}}(\mathsf{o}_1^{\mathcal{I}}, \mathsf{o}_3^{\mathcal{I}}) \geq \sup\{\ldots, \min(0.8, 0.9), \ldots\} \geq 0.8$*. Furthermore, from* $(\mathsf{isAPartOf}^-)^{\mathcal{I}}(\mathsf{o}_3^{\mathcal{I}}, \mathsf{o}_1^{\mathcal{I}}) \geq 0.8$ *and* $\mathsf{isAPartOf}^- \sqsubseteq \mathsf{hasAPart}$ *we deduce that,* $\mathsf{hasAPart}^{\mathcal{I}}(\mathsf{o}_3^{\mathcal{I}}, \mathsf{o}_1^{\mathcal{I}}) \geq 0.8$ *and finally,* $(\exists\mathsf{hasAPart.OPIFG})^{\mathcal{I}}(\mathsf{o}_3^{\mathcal{I}}) = \sup\{\ldots, \min(\mathsf{hasAPart}^{\mathcal{I}}(\mathsf{o}_3^{\mathcal{I}}, \mathsf{o}_1^{\mathcal{I}}), (\mathsf{OPIFG})^{\mathcal{I}}(\mathsf{o}_1^{\mathcal{I}})), \ldots\} \geq 0.75$.

$\diamondsuit$

Table 3.2: Conjugated pairs of fuzzy assertions

	$\phi < m$	$\phi \leq m$
$\phi \geq n$	$n \geq m$	$n > m$
$\phi > n$	$n \geq m$	$n \geq m$

Following [Str01], we introduce the concept of conjugated pairs of fuzzy assertions to represent pairs of assertions that form a contradiction. The possible conjugated pairs are defined in Table 3.2, where ϕ represents a crisp assertion of the form $a : C$ or $\langle a, b\rangle : R$. So for example, the two fuzzy assertions $(a : C) > 0.7$ and $(a : C) \leq 0.7$ conjugate since this case falls under the cell of the second line and first column of Table 3.2. In the presence of inverse roles and role hierarchies, the definition of conjugated role assertions has to be extended. We say that two role assertions $\phi \geq n_1$ and $\psi \leq n_2$ conjugate if $\phi = (a, b) : R$, $\psi = (b, a) : \mathsf{Inv}(R)$ and $n_1 > n_2$. Similarly for the rest of the inequalities. Finally, regarding role hierarchies, one should also take under consideration possible sub- or super-roles when checking for conjugation two fuzzy assertions that involve roles. For example, the fuzzy assertion $(\langle a, b\rangle : R) > 0.6$, conjugates with $(\langle a, b\rangle : S) \leq 0.5$, provided that $R \sqsubseteq^* S$.

Theorem 3 *Fuzzy interpretations coincide with crisp interpretations if we restrict to the membership degrees of 0 and 1.*

As we explained in Chapter 2 different choices of fuzzy triples and quadruples defines different fuzzy logics. Hence, this justifies the need for introducing a special notation for distinguishing between fuzzy DLs that use different norm functions. More precisely, the notation f$_{\mathcal{J}}$-DL, where $\mathcal{J}$ is a fuzzy implication function, has been proposed in the literature [SST$^+$05b] to indicate that the fuzzy DL uses the specified $\mathcal{J}$ operation to perform fuzzy implication. The rest of fuzzy operations are defined by the fuzzy implication [Haj98, KN99]. For example the notation f$_{KD}$-$\mathcal{SHOIN}$ indicates the fuzzy $\mathcal{SHOIN}$ language which uses the Lukasiewicz complement, the Gödel t-norm and t-conorm and the Kleene-Dienes fuzzy implication.

3.2 Inference Problems of fuzzy DLs

In the current section we will present the inference problems of f-DLs. Moreover, we will show how to deal with GCIs in the context of fuzzy DLs. Until recently this was considered an open problem.

An f-$\mathcal{SHOIQ}$ knowledge base Σ is *satisfiable* (*unsatisfiable*) iff there exists (does not exist) a fuzzy interpretation $\mathcal{I}$ which satisfies all axioms in Σ. An f-$\mathcal{SHOIQ}$-concept C is *n-satisfiable* w.r.t. Σ iff there exists a model $\mathcal{I}$ of Σ for which there is some $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) = n$, and $n \in (0,1]$; C subsumes D w.r.t. Σ iff for every model $\mathcal{I}$ of Σ we have $\forall d \in \Delta^{\mathcal{I}}, C^{\mathcal{I}}(d) \leq D^{\mathcal{I}}(d)$; a fuzzy ABox $\mathcal{A}$ is *consistent* (*inconsistent*) w.r.t. a fuzzy TBox $\mathcal{T}$ and RBox $\mathcal{R}$ if there exists (does not exist) a model $\mathcal{I}$ of $\mathcal{T}$ and $\mathcal{R}$ that satisfies each assertion in $\mathcal{A}$. Given a fuzzy concept axiom, a fuzzy role axiom or a fuzzy assertion ϕ, Σ *entails* ϕ, written $\Sigma \models \phi$, iff for all models $\mathcal{I}$ of Σ, $\mathcal{I}$ satisfies ϕ.

Let $\Sigma = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$, be a fuzzy knowledge base. It has been proved that all inference problems of f-DLs can be reduced to ABox consistency w.r.t. $\mathcal{T}$ and $\mathcal{R}$. More precisely, C is n-satisfiable w.r.t. Σ iff $\langle \mathcal{T}, \mathcal{R}, \{(\mathsf{a} : C) \geq n\} \rangle$ is satisfiable, $\Sigma \models \phi \bowtie n$ iff $\Sigma = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{\phi \neg \bowtie n\} \rangle$ is unsatisfiable and $\Sigma \models C \sqsubseteq D$ iff $\Sigma = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{(\mathsf{a} : C) \geq n, (\mathsf{a} : D) < n\} \rangle$, for both $n \in \{n_1, n_2\}$, $n_1 \in (0, 0.5]$ and $n_2 \in (0.5, 1]$, is unsatisfiable [Str01]. In the past, the consistency problem in f-DLs has been considered with respect to a simple and acyclic TBox. Only recently a procedure for deciding fuzzy ABox consistency w.r.t. general and/or cyclic TBoxes has been developed [SSSP06]. In classical DLs general and cyclic TBoxes are handled by a process called *internalization* [HS99]. The method of internalization is based on the law of excluded middle, which is not always satisfied in f-DLs. In [SSSP06] the authors use simple case analysis to provide the following result,

Lemma 4 *[SSSP06] A fuzzy interpretation $\mathcal{I}$ satisfies $C \sqsubseteq D$ iff for all $n \in [0,1]$ and $a \in \Delta^{\mathcal{I}}$, either $C^{\mathcal{I}}(a) < n$ or $D^{\mathcal{I}}(a) \geq n$.*

The above lemma suggests that the models represented by a fuzzy TBox $\mathcal{T}$ can be encoded in the form of mutually exclusive fuzzy ABoxes. For example, if $\mathcal{T} = \{C_1 \sqsubseteq D_1, C_2 \sqsubseteq D_2\}$, then the alternatives, $\{\langle a : C_1 < n\rangle, \langle a : C_2 < n\rangle\}$ or $\{\langle a : C_1 < n\rangle, \langle a : C_2 \geq n\rangle\}$ or $\{\langle a : C_1 \geq n\rangle, \langle a : C_2 < n\rangle\}$ or $\{\langle a : C_1 \geq n\rangle, \langle a : C_2 \geq n\rangle\}$ for all $n \in [0,1]$, are created and capture exactly the models represented by the fuzzy TBox. These alternatives should be asserted for all individuals $a \in \mathbf{I}$ and for all values $n \in [0,1]$. Observe that this is quite similar to the case of crisp DLs where for the above TBox the internalization method would create the concept $(\neg C_1 \sqcup D_1) \sqcap (\neg C_2 \sqcup D_2)$, which represents four different possibilities. In fuzzy DLs we additionally have to perform case analysis for each membership degree n.

However, it is practically impossible to devise a terminating reasoning algorithm that uses Lemma 4 to handle GCIs and cyclic axioms as we cannot realistically apply it to all $n \in [0,1]_{\mathbb{Q}}$. Fortunately, we can restrict these n to a *finite* set of values. Indeed, a good candidate is the set $N^{\mathcal{A}}$, defined as $N^{\mathcal{A}} = X^{\mathcal{A}} \cup \{1 - n \mid n \in X^{\mathcal{A}}\}$, where $X^{\mathcal{A}} = \{0, 0.5, 1\} \cup \{n \mid \langle \alpha \bowtie n\rangle \in \mathcal{A}\}$. Intuitively, this means that if a f$_{KD}$-$\mathcal{ALC}$ ABox is consistent, then there exists a model where the membership degrees used to build the model are restricted to those that exist in the ABox. For instance, in order to satisfy $\{\langle a{:}C \geq n\rangle\}$, we set $C^{\mathcal{I}}(a^{\mathcal{I}}) = n$, while to satisfy $\{\langle a{:}C > n\rangle\}$, we set $C^{\mathcal{I}}(a^{\mathcal{I}}) = n + \epsilon$, for a sufficiently small $\epsilon \in [0,1]$.

In the following, we assume that an ABox $\mathcal{A}$ has been *normalized*, i.e. fuzzy assertions of the form $\langle a{:}C > n\rangle$ are replaced by $\langle a{:}C \geq n + \epsilon\rangle$ and those of the form $\langle a{:}C < n\rangle$, by $\langle a{:}C \leq n - \epsilon\rangle$. Please note that in a normalized fuzzy KB we allow the degree to range in $[-\epsilon, 1+\epsilon]_{\mathbb{Q}}$ in place of $[0,1]_{\mathbb{Q}}$. It can be proved that the process of normalization is satisfiability preserving.

Proposition 1 *Let $\Sigma = \langle \mathcal{T}, \mathcal{A}\rangle$ be a fuzzy knowledge base. Then Σ is satisfiable if and only if its normalized variant is satisfiable.*

3.3 Reasoning in f$_{KD}$-$\mathcal{SHOIN}$

Reasoning in DLs is usually performed with tableaux decision procedures [BDS93]. Such procedures try to prove the consistency of an Abox $\mathcal{A}$ by attempting to construct a model for it. Since concepts that appear in assertions in $\mathcal{A}$ might be complex concept descriptions, such algorithms apply *expansion rules*, which decompose the initial concept, to sub-concepts, until no rule is applicable or an evident contradiction (*clash*) is reached. Proceeding that way leads to the creation of a model for $\mathcal{A}$, which has a graph-like shape [HS05]. The nodes in such a graph correspond to objects in the model, and edges to certain relations that connect two nodes. Each node x is labelled with a set of concepts $\mathcal{L}(x)$, and each edge $\langle x, y\rangle$ with a set of roles $\mathcal{L}(\langle x, y\rangle)$. In the fuzzy case, since we have fuzzy assertions, we extend these sets to also include the membership degree that a node belongs to a concept as well as the type of inequality that holds for the fuzzy assertion, thus speaking of *membership triples*. For example a fuzzy assertion of the form $\langle a : C \geq n\rangle$ is represented with a node x_a, labelled with the set, $\mathcal{L}(x_a) = \{\langle C, \geq, n\rangle\}$.

In [SST$^+$05b, SST$^+$05a] tableaux decision procedures for deciding the consistency of f$_{KD}$-$\mathcal{SI}$ and f$_{KD}$-$\mathcal{SHIN}$ ABoxes has been presented. Since we argue that nominals should not be fuzzyfied, this algorithm, together with the results obtained in [HS05] for crisp $\mathcal{SHOIN}$, can be extended to provide a tableaux procedure for f$_{KD}$-$\mathcal{SHOIN}$. The only additional non-standard rules that are needed are those which would ensure that membership triples involving nominal concepts $\{\mathrm{o}\}$ would contain a membership degree of either 0 or 1, in order to respect the semantics of nominal concepts. In the following we first give the definition of an f$_{KD}$-$\mathcal{SHIN}$ completion-forest as well as the tableaux expansion rules. This definition and rules extend our results about f$_{KD}$-$\mathcal{SI}$ completion-trees and tableaux expansion rules, previously reported in deliverable D2.5.3. In the following we present the algorithm for the f$_{KD}$-$\mathcal{SHIN}$ DL and then we extend it with those expansions rules that are needed to provide reasoning support for nominals and GCIs.

Definition 5 *A completion-forest $\mathcal{F}$ for an f$_{KD}$-$\mathcal{SHIN}$ ABox $\mathcal{A}$ is a collection of trees whose distinguished roots are arbitrarily connected by edges. Each node x is labelled with a set $\mathcal{L}(x) = \{\langle C, \bowtie, n\rangle\}$, where $C \in sub(\mathcal{A})$ and $n \in [0,1]$. Each edge $\langle x, y\rangle$ is labelled with a set $\mathcal{L}(\langle x, y\rangle) = \{\langle R, \bowtie, n\rangle\}$, where $R \in \boldsymbol{R}_{\mathcal{A}}$ are (possibly inverse) roles occurring in $\mathcal{A}$. Intuitively, each triple $\langle C, \bowtie, n\rangle$ ($\langle R, \bowtie, n\rangle$), called* membership triple, *represents the membership degree and the type of assertion of each node (pair of nodes) to a concept $C \in sub(\mathcal{A})$ (role $R \in \boldsymbol{R}_{\mathcal{A}}$).*

*If nodes x and y are connected by an edge $\langle x, y\rangle$ with $\langle P, \bowtie, n\rangle \in \mathcal{L}(\langle x, y\rangle)$, and $P \,\underline{\circledast}\, R$, then y is called an $R_{\bowtie n}$-*successor *of x and x is called an $R_{\bowtie n}$-*predecessor *of y. If y is an $R_{\bowtie n}$-successor or an $\mathsf{Inv}(R)_{\bowtie n}$-*predecessor *of x, then y is called an $R_{\bowtie n}$-neighbour of x. Let y be an $R_{>n}$-neighbour of x, the edge $\langle x, y\rangle$ is* conjugated *with triples $\langle R, \lhd, m\rangle$ if $n \geq m$. Similarly, we can extend it to the cases of $R_{\geq n}$-, $R_{<n}$- and $R_{\leq n}$-neighbours. A node x is an R-successor (resp. R-predecessor or R-neighbour) of y if it is an $R_{\bowtie n}$-successor (resp. $R_{\bowtie n}$-predecessor or $R_{\bowtie n}$-neighbour) of y for some role R. As usual,* ancestor *is the transitive closure of* predecessor.

For two roles P and R, a node x in $\mathcal{F}$, two inequality types $\bowtie$, $\bowtie'$ and a membership degree $n \in [0,1]$ we define: $R_C^{\mathcal{F}}(x, \bowtie, \bowtie', n) = \{y \mid y$ is an $R_{\bowtie n'}$-neighbour, and $\langle x, y\rangle$ is conjugated with $\langle P, \bowtie', n\rangle$, with $P \,\underline{\circledast}\, R\}$.

A node x is blocked *iff it is not a root node and it is either directly or indirectly blocked. A node x is* directly blocked *iff none of its ancestors is blocked, and it has ancestors x', y and y' such that:*

1. *y is not a root node,*
2. *x is a successor of x' and y a successor of y',*
3. *$\mathcal{L}(x) = \mathcal{L}(y)$ and $\mathcal{L}(x') = \mathcal{L}(y')$ and,*

Table 3.3: Tableaux expansion rules for f_{KD}-$\mathcal{SI}$

Rule		Description
$(\neg)$	if 1.	$\langle \neg C, \bowtie, n\rangle \in \mathcal{L}(x)$
	2.	and $\langle C, \bowtie^{-}, 1-n\rangle \notin \mathcal{L}(x)$
	then	$\mathcal{L}(x) \to \mathcal{L}(x) \cup \{\langle C, \bowtie^{-}, 1-n\rangle\}$
$(\sqcap_{\rhd})$	if 1.	$\langle C_1 \sqcap C_2, \rhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, and
	2.	$\{\langle C_1, \rhd, n\rangle, \langle C_2, \rhd, n\rangle\} \not\subseteq \mathcal{L}(x)$
	then	$\mathcal{L}(x) \to \mathcal{L}(x) \cup \{\langle C_1, \rhd, n\rangle, \langle C_2, \rhd, n\rangle\}$
$(\sqcup_{\lhd})$	if 1.	$\langle C_1 \sqcup C_2, \lhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, and
	2.	$\{\langle C_1, \lhd, n\rangle, \langle C_2, \lhd, n\rangle\} \not\subseteq \mathcal{L}(x)$
	then	$\mathcal{L}(x) \to \mathcal{L}(x) \cup \{\langle C_1, \lhd, n\rangle, \langle C_2, \lhd, n\rangle\}$
$(\sqcup_{\rhd})$	if 1.	$\langle C_1 \sqcup C_2, \rhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, and
	2.	$\{\langle C_1, \rhd, n\rangle, \langle C_2, \rhd, n\rangle\} \cap \mathcal{L}(x) = \emptyset$
	then	$\mathcal{L}(x) \to \mathcal{L}(x) \cup \{C\}$ for some $C \in \{\langle C_1, \rhd, n\rangle, \langle C_2, \rhd, n\rangle\}$
$(\sqcap_{\lhd})$	if 1.	$\langle C_1 \sqcap C_2, \lhd, n\rangle \mathcal{L}(x)$, x is not indirectly blocked, and
	2.	$\{\langle C_1, \lhd, n\rangle, \langle C_2, \lhd, n\rangle\} \cap \mathcal{L}(x) = \emptyset$
	then	$\mathcal{L}(x) \to \mathcal{L}(x) \cup \{C\}$ for some $C \in \{\langle C_1, \lhd, n\rangle, \langle C_2, \lhd, n\rangle\}$
$(\exists_{\rhd})$	if 1.	$\langle \exists R.C, \rhd, n\rangle \in \mathcal{L}(x)$, x is not blocked,
	2.	x has no $R_{\rhd n}$-neighbour y with $\langle C, \rhd, n\rangle \in \mathcal{L}(y)$
	then	create a new node y with $\mathcal{L}(\langle x, y\rangle) = \{\langle R, \rhd, n\rangle\}$, $\mathcal{L}(y) = \{\langle C, \rhd, n\rangle\}$,
$(\forall_{\lhd})$	if 1.	$\langle \forall R.C, \lhd, n\rangle \in \mathcal{L}(x)$, x is not blocked,
	2.	x has no $R_{\lhd^{-} 1-n}$-neighbour y with $\langle C, \lhd, n\rangle \in \mathcal{L}(y)$
	then	create a new node y with $\mathcal{L}(\langle x, y\rangle) = \{\langle R, \lhd^{-}, 1-n\rangle\}$, $\mathcal{L}(y) = \{\langle C, \lhd, n\rangle\}$,
$(\forall_{\rhd})$	if 1.	$\langle \forall R.C, \rhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, and
	2.	x has an $R_{\rhd' n_1}$-neighbour y with $\langle C, \rhd, n\rangle \notin \mathcal{L}(y)$ and
	3.	$\langle R, \rhd^{-}, 1-n\rangle$ is conjugated with the edge $\langle x, y\rangle$
	then	$\mathcal{L}(y) \to \mathcal{L}(y) \cup \{\langle C, \rhd, n\rangle\}$,
$(\exists_{\lhd})$	if 1.	$\langle \exists R.C, \lhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked and
	2.	x has an $R_{\rhd n_1}$-neighbour y with $\langle C, \lhd, n\rangle \notin \mathcal{L}(y)$ and
	3.	$\langle R, \lhd, n\rangle$ is conjugated with the edge $\langle x, y\rangle$
	then	$\mathcal{L}(y) \to \mathcal{L}(y) \cup \{\langle C, \lhd, n\rangle\}$,
$(\forall_{+})$	if 1.	$\langle \forall S.C, \rhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, and
	2.	there is some R, with $\mathsf{Trans}(R)$, and $R \underline{\boxast} S$, x has a $R_{\rhd' n_1}$-neighbour y with, $\langle \forall R.C, \rhd, n\rangle \notin \mathcal{L}(y)$, and
	3.	$\langle R, \rhd^{-}, 1-n\rangle$ is conjugated with the edge $\langle x, y\rangle$
	then	$\mathcal{L}(y) \to \mathcal{L}(y) \cup \{\langle \forall R.C, \rhd, n\rangle\}$,
$(\exists_{+})$	if 1.	$\langle \exists S.C, \lhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked and
	2.	there is some R, with $\mathsf{Trans}(R)$, and $R \underline{\boxast} S$, x has a $R_{\rhd n_1}$-neighbour y with, $\langle \exists R.C, \lhd, n\rangle \notin \mathcal{L}(y)$, and
	3.	$\langle R, \lhd, n\rangle$ is conjugated with the edge $\langle x, y\rangle$
	then	$\mathcal{L}(y) \to \mathcal{L}(y) \cup \{\langle \exists R.C, \lhd, n\rangle\}$,

4. $\mathcal{L}(\langle x', x\rangle) = \mathcal{L}(\langle y', y\rangle)$.

In this case we say that y blocks x. A node y is indirectly blocked iff one of its ancestors is blocked, or it is a successor of a node x and $\mathcal{L}(\langle x, y\rangle) = \emptyset$.

For a node x, $\mathcal{L}(x)$ is said to contain a clash if it contains one of the following:

- *two conjugated pairs of triples,*

Table 3.4: Tableaux rules for number restrictions

Rule		Description
$(\geq_{\rhd})$	if 1.	$\langle \geq pR, \rhd, n\rangle \in \mathcal{L}(x)$, x is not blocked,
	2.	there are no p $R_{\rhd n}$-neighbours $y_1, \ldots, y_p$ of x
	3.	with $y_i \neq y_j$ for $1 \leq i < j \leq p$
	then	create p new nodes $y_1, \ldots, y_p$, with $\mathcal{L}(\langle x, y_i\rangle) = \{\langle R, \rhd, n\rangle\}$ and $y_i \neq y_j$ for $1 \leq i < j \leq p$
$(\leq_{\lhd})$	if 1.	$\langle \leq pR, \lhd, n\rangle \in \mathcal{L}(x)$, x is not blocked,
	then	apply $(\geq_{\rhd})$-rule for the triple $\langle \geq (p+1)R, \lhd^-, 1-n\rangle$
$(\leq_{\rhd})$	if 1.	$\langle \leq pR, \rhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked,
		$\sharp R_C^{\mathcal{F}}(x, \rhd', \rhd^-, 1-n) > p$, there are two of them y, z, with no $y \neq z$ and
	3.	y is neither a root node nor an ancestor of z
	then	1. $\mathcal{L}(z) \to \mathcal{L}(z) \cup \mathcal{L}(y)$ and
		2. if z is an ancestor of x
		then $\mathcal{L}(\langle z, x\rangle) \longrightarrow \mathcal{L}(\langle z, x\rangle) \cup \mathsf{Inv}(\mathcal{L}(\langle x, y\rangle))$
		else $\mathcal{L}(\langle x, z\rangle) \longrightarrow \mathcal{L}(\langle x, z\rangle) \cup \mathcal{L}(\langle x, y\rangle)$
		3. $\mathcal{L}(\langle x, y\rangle) \longrightarrow \emptyset$
		4. Set $u \neq z$ for all u with $u \neq y$
$(\geq_{\lhd})$	if 1.	$\langle \geq pR, \lhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked,
	then	apply $(\leq_{\rhd})$-rule for the triple $\langle \leq (p-1)R, \lhd^-, 1-n\rangle$
$(\leq_{r_{\rhd}})$	if 1.	$\langle \leq pR, \rhd, n\rangle \in \mathcal{L}(x)$,
		$\sharp R_C^{\mathcal{F}}(x, \rhd', \rhd^-, 1-n) > p$, there are two of them y, z, both root nodes, with no $y \neq z$ and
	then	1. $\mathcal{L}(z) \to \mathcal{L}(z) \cup \mathcal{L}(y)$ and
		2. For all edges $\langle y, w\rangle$:
		i. if the edge $\langle z, w\rangle$ does not exist, create it with $\mathcal{L}(\langle z, w\rangle) = \emptyset$
		ii. $\mathcal{L}(\langle z, w\rangle) \longrightarrow \mathcal{L}(\langle z, w\rangle) \cup \mathcal{L}(\langle y, w\rangle)$
		3. For all edges $\langle w, y\rangle$:
		i. if the edge $\langle w, z\rangle$ does not exist, create it with $\mathcal{L}(\langle w, z\rangle) = \emptyset$
		ii. $\mathcal{L}(\langle w, z\rangle) \longrightarrow \mathcal{L}(\langle w, z\rangle) \cup \mathcal{L}(\langle w, y\rangle)$
		4. Set $\mathcal{L}(y) = \emptyset$ and remove all edges to/from y
		5. Set $u \neq z$ for all u with $u \neq y$ and set $y \doteq z$
$(\geq_{r_{\lhd}})$	if 1.	$\langle \geq pR, \lhd, n\rangle \in \mathcal{L}(x)$,
	then	apply $(\leq_{r_{\rhd}})$-rule for the triple $\langle \leq (p-1)R, \lhd^-, 1-n\rangle$

- *one of the triples* $\langle \bot, \geq, n\rangle$, $\langle \top, \leq, n\rangle$, *with* $n > 0$, $n < 1$, $\langle \bot, >, n\rangle$, $\langle \top, <, n\rangle$ $\langle C, <, 0\rangle$ *or* $\langle C, >, 1\rangle$,

- *some triple* $\langle \leq pR, \rhd, n\rangle$ *and* x *has* $p+1$ $R_{\rhd' n_i}$*-neighbours* $y_0, \ldots, y_p$ *conjugated with* $\langle R, \rhd^-, 1-n\rangle$ *and* $y_i \neq y_j$, $n_i, n \in [0,1]$, *for all* $0 \leq i < j \leq p$, *or*

- *some triple* $\langle \geq pR, \lhd, n\rangle$ *and* x *has* p $R_{\rhd n_i}$*-neighbours* $y_0, \ldots, y_{p-1}$, *conjugated with* $\langle R, \lhd, n\rangle$ *and* $y_i \neq y_j$, $n_i, n \in [0,1]$, *for all* $0 \leq i < j \leq p-1$.

For an f_{KD}*-*$\mathcal{SHIN}$ *ABox* $\mathcal{A}$*, the algorithm initialises a forest* $\mathcal{F}$ *to contain (i) a root node* x_0^i*, for each individual* $a_i \in \mathbf{I}_{\mathcal{A}}$ *occurring in the ABox* $\mathcal{A}$*, labelled with* $\mathcal{L}(x_0^i)$ *such that* $\{\langle C_i, \rhd, n\rangle\} \subseteq \mathcal{L}(x_0^i)$ *for each assertion of the form* $(a_i : C_i) \rhd n \in \mathcal{A}$*, (ii) an edge* $\langle x_0^i, x_0^j\rangle$*, for each assertion* $(\langle a_i, a_j\rangle : R_i) \rhd n \in \mathcal{A}$*, labelled with* $\mathcal{L}(\langle x_0^i, x_0^j\rangle)$ *such that* $\{\langle R_i, \rhd, n\rangle\} \subseteq \mathcal{L}(\langle x_0^i, x_0^j\rangle)$*, (iii) the relation* $\neq$ *as* $x_0^i \neq x_0^j$ *if* $a_i \neq a_j \in \mathcal{A}$ *and the relation* $\doteq$ *to be empty. Finally, the algorithm expands* $\mathcal{R}$ *by adding role inclusion axioms*

$\mathsf{Inv}(P) \sqsubseteq \mathsf{Inv}(R)$, *for all* $P \sqsubseteq R \in \mathcal{R}$ *and by adding* $\mathsf{Trans}(\mathsf{Inv}(R))$ *for all* $\mathsf{Trans}(R) \in \mathcal{R}$. *$\mathcal{F}$ is then expanded by repeatedly applying the completion rules from Tables 3.3 and 3.4. The completion-forest is complete when, for some node x, $\mathcal{L}(x)$ contains a clash, or none of the completion rules is applicable. The algorithm stops when a clash occurs; it answers '$\mathcal{A}$ is consistent w.r.t. $\mathcal{R}$' iff the completion rules can be applied in such a way that they yield a complete and clash-free completion-forest, and '$\mathcal{A}$ is inconsistent w.r.t. $\mathcal{R}$' otherwise.*

Table 3.5: The new expansion rules for f$_{KD}$-$\mathcal{SHOIN}$

Rule		Description
$\{o\}_{\triangleright}$	if 1.	$\langle\{o\}, \triangleright, n\rangle \in \mathcal{L}(x)$, and
	2.	$\langle\{o\}, \geq, 1\rangle \notin \mathcal{L}(x)$
	then	$\mathcal{L}(x) \cup \{\langle\{o\}, \geq, 1\rangle\}$
$\{o\}_{\triangleleft}$	if 1.	$\langle\{o\}, \triangleleft, n\rangle \in \mathcal{L}(x)$, and
	2.	$\langle\{o\}, \leq, 0\rangle \notin \mathcal{L}(x)$
	then	$\mathcal{L}(x) \cup \{\langle\{o\}, \leq, 0\rangle\}$
$(C \sqsubseteq D)$	if 1.	$C \sqsubseteq D \in \mathcal{T}$ and
	2.	$\{\langle C, \leq, n-\epsilon\rangle, \langle D, \geq, n\rangle\} \cap \mathcal{L}(x) = \emptyset$ for $n \in N^{\mathcal{A}}$
	then	$\mathcal{L}(x) \rightarrow \mathcal{L}(x) \cup \{E\}$ for some $E \in \{\langle C, \leq, n-\epsilon\rangle, \langle D, \geq, n\rangle\}$

The above definition gives us a tableaux decision procedure for the fuzzy DL language f$_{KD}$-$\mathcal{SHIN}$. In order to extend the algorithm to handle nominals we have to take under consideration all the notions introduced in [HS05] for crisp $\mathcal{SHOIN}$. In fact most notions, like blockable nodes, merging, pruning and priority of rule expansion are the same. The only additions to the algorithm are those that will ensure the interpretation of the nominal concepts as crisp concepts, since this is how we have interpreted them. In Table 3.5 we can see these rules together with the rule that is intended for reasoning with GCIs. As we see the rule for GCIs is based on the notion of a normalized KB that we introduced in the previous section.

Regarding implementation issues we have implemented a Fuzzy Reasoning Engine (FiRE)[2] for the f$_{KD}$-$\mathcal{SHIN}$ language, based on the direct tableaux rules in [SST$^+$05a], and we have started the extension of the algorithm to cover the f$_{KD}$-$\mathcal{SHOIN}$ language with general concept inclusion axioms; optimization capabilities are also investigated. Please note that how to reason with other norm operations in expressive fuzzy DLs, like fuzzy $\mathcal{SI}$ or $\mathcal{SHIN}$, remains an open problem.

[2]FiRE is available at `http://www.image.ece.ntua.gr/~nsimou`

3.4 FiRE: A Fuzzy Reasoning Engine

FiRE (Fuzzy Reasoning Engine) [SSSK06] is a prototype implementation of a reasoning algorithm for a very expressive fuzzy Description Logic language, namely f_{KD}-$\mathcal{SHIN}$ [SST$^+$05a]. This algorithm builds upon previous results about reasoning with fuzzy Description Logics, by extending the reasoning algorithm of f_{KD}-$\mathcal{ALC}$ [Str01] to handle most of the features of OWL. Currently FiRE supports only simple TBoxes, i.e. no General Concept Inclusion axioms are allowed. Figure 3.1 illustrates the graphical interface of the FiRE platform. As we can see FiRE is formed by three different components. The first one, which is in the upper left part of the user interface, is the editor panel; the second, on the upper upper right part, is the inference services panel; the last, on the bottom consists of a set of output tabs.

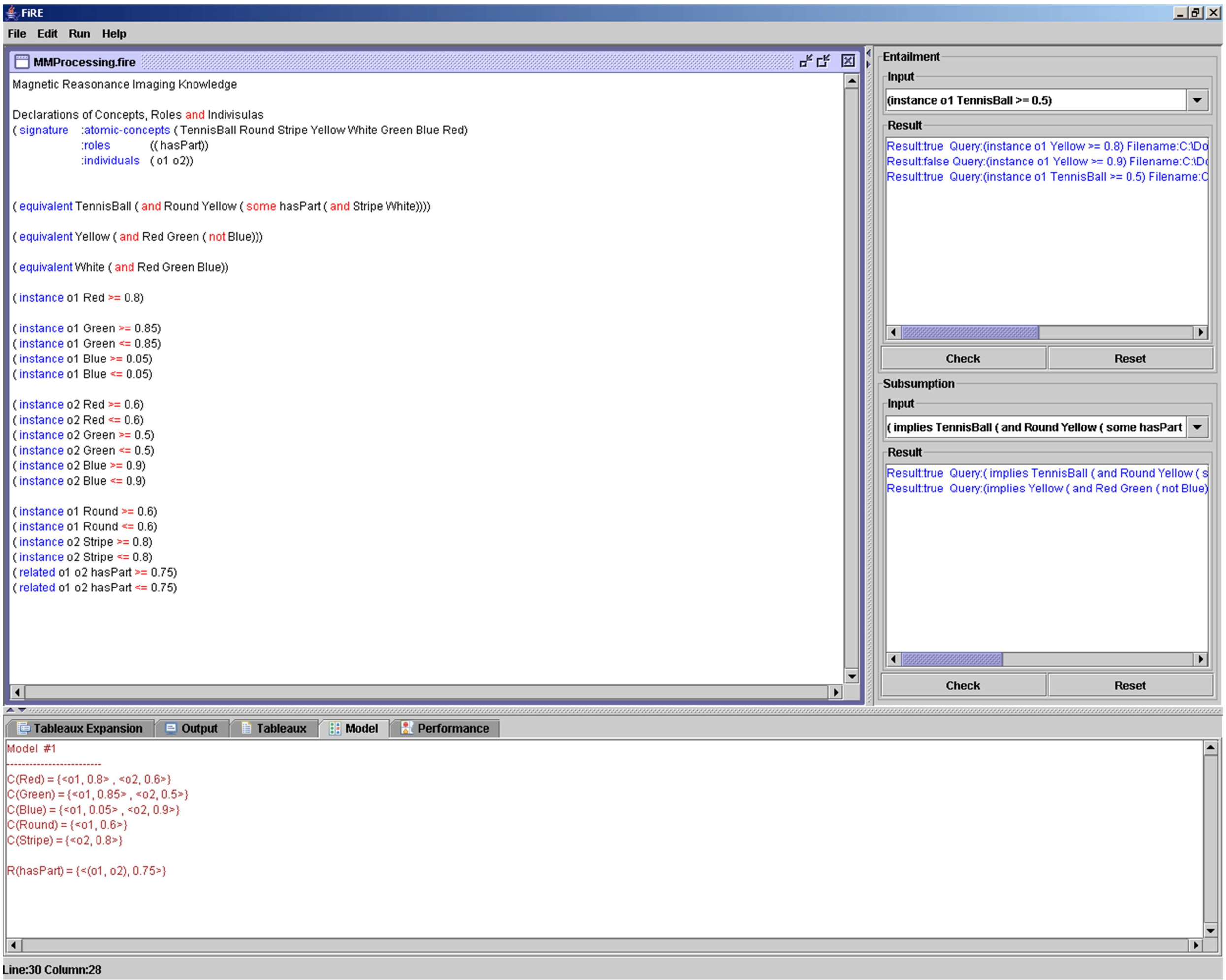

Figure 3.1: Screen Shot of FiRE User Interface

3.4.1 Editor

In the editor area one is able to open and edit or create from scratch a new knowledge base. FiRE uses the same syntax as the RACER (http://www.sts.tu-harburg.de/ r.f.moeller/racer/) Description Logic reasoning engine in order to encode the captured knowledge. Obviously, in FiRE we had to slightly extend the syntax of RACER in order to support fuzzy facts.

As we can see in Figure 3.1, by using the keyword *equivalent* we can provide the definitions for the fuzzy concepts TenniBall, White and Yellow. Finally, using the keywords *instance* and *related* as well as an inequality type and a membership degree we can define fuzzy facts. In addition to the fuzzy facts introduced in the previous section, we have specified that segment o_2 is red to a degree 0.6, green to a degree 0.5, blue to a degree 0.9 and its shape represents a stripe to a degree 0.8. Furthermore, we have extended our knowledge about segment o_1 saying that it is round to a degree 0.6.

3.4.2 Inference Services

The FiRE platform supports three types of inferences. The first one is the ability to check the consistency of a fuzzy knowledge base. The other two inferences are more focused on querying the given knowledge in order to derive new implied knowledge. To provide reasoning support for f-OWL we have reduced a f-OWL ontology to a fuzzy DL knowledge base [SST+05c] in a manner similar to the way OWL is reduced to DLs.

The first type of query that is supported is the entailment of a fuzzy fact. This functionality is provided in the upper part of the inference services panel of the FiRE platform as it is depicted in Figure 3.1. For example one useful query to our fuzzy knowledge would be to ask if segment o_1 is yellow to a degree greater or equal than 0.8. The user can input this query in the upper part of the inference services panel by using the RACER syntax. For our fuzzy knowledge the answer is positive since as we explained in the previous section the semantics of the language entail such a fact. Another fuzzy fact that our knowledge entails is that o_1 is a tennis ball to a degree greater or equal than 0.5.

The last inference service provided is the subsumption between two fuzzy concepts. This is provided in the lower part of the inference services panel, as it can be seen in Figure 3.1. For example we can query if concept Yellow is a sub-concept of the conjunction of the fuzzy concepts Red, Green and not Blue which is obviously true. In RACER syntax subsumption is specified with the keyword *implies*.

3.4.3 Output Tabs

Finally FiRE uses a number of output tabs to provide information about the fuzzy knowledge base. In Figure 3.1 the *model* output tab has been selected. This tab returns the model (fuzzy interpretation) that satisfies the concept, role and instance axioms specified in the fuzzy knowledge base if they are consistent. Figure 3.1 shows a model of the fuzzy knowledge base after we have checked its consistency. We can see that the concept Red is a fuzzy concept with object o_1 belonging to a degree 0.8 and object o_2 belonging to a degree 0.6. hasPart is a fuzzy property where the pair (o_1, o_2) belongs to a degree 0.75; similarly with the rest of the fuzzy concepts. Another important tab are the tableaux expansion tab where one is able to see a trace of the application of the reasoning algorithm. Other tabs that are provided is the tableaux tab where we can see the structure that the reasoning algorithm has created, the output tab which provides a view of the initial fuzzy knowledge and the performance tab which provides a view of information about the usage of computing resources.

Regarding the efficiency of the implementation, we have to mention that currently FiRE does not support any of the optimization techniques of classical DL. Hence, the behavior of the implementation is dependent on the form and size of the input knowledge base. Roughly speaking FiRE operates well with small and medium size knowledge bases (about 100 concepts), and which do not contain concept which use complex constructors (speaking in terms of sources of complexity), like disjointness constructors and at-most restrictions which introduce non-determinism. Some first investigations have shown that there is a lot of room for optimization and that most of the classical DL optimization methods would be applicable to fuzzy DLs with some necessary adaptations.

Chapter 4

Fuzzy OWL

In the current chapter we will present a fuzzy extension to the OWL language, thus creating the fuzzy OWL language. Additionally, we will present a translation technique which reduces an f-OWL ontology to a f-$\mathcal{SHOIN}$ knowledge base. Our presentation follows the one in [SST$^+$05c], but provides some revisions to the semantics of the language, as well as the reduction technique. Moreover, we also address the issue of the RDF/XML syntax of f-OWL that has not been previously presented.

4.1 Semantics of fuzzy OWL

Our fuzzy OWL language uses essentially the same syntax as the crisp OWL language [BvHH$^+$04]. Hence, one is able to create OWL class descriptions and OWL class and property axioms in exactly the same way this is done in OWL. For example one can define the class of white things as the set of things that are red, green and blue at the same time. In OWL abstract syntax [PSHH04] this definition could be written as, Class(White complete intersectionOf(Red Green Blue)).

The differences between crisp OWL and fuzzy OWL arise in the definition of facts (individual axioms). In the case of f-OWL besides specifying the membership of an individual (pair of individuals) to a class (property) we also need to specify the membership degree that this individual (pair of individuals) belongs to the class (property), and an inequality, as in the case of f-$\mathcal{SHOIN}$ fuzzy assertions. Such axioms are referred to as *fuzzy facts*. For example, one might want to state that an image region, reg_1, is blue to a degree greater or equal than 0.8. As we will see in the following, in f-OWL the abstract syntax of such an axiom is Individual(reg_1 type(Blue) $>= 0.8$).

Although the syntax modifications are minor, the semantics of f-OWL are based on fuzzy interpretations, and interpret OWL classes and properties as fuzzy sets and fuzzy relations. In the case of f-OWL DL (that we are studying here) these interpretations are fairly standard by description logic standards. Hence, a fuzzy interpretation is a pair $\mathcal{I} =$

Table 4.1: Fuzzy OWL Class Descriptions

Abstract Syntax	DL Syntax	Semantics
Class(A)	A	$A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$
Class(owl:Thing)	$\top$	$\top^{\mathcal{I}}(a) = 1$
Class(owl:Nothing)	$\bot$	$\bot^{\mathcal{I}}(a) = 0$
intersectionOf($C, D, \dots$)	$C \sqcap D$	$(C \sqcap D)^{\mathcal{I}}(a) = t(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$
unionOf($C, D, \dots$)	$C \sqcup D$	$(C \sqcup D)^{\mathcal{I}}(a) = u(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$
complementOf(C)	$\neg C$	$(\neg C)^{\mathcal{I}}(a) = c(C^{\mathcal{I}}(a))$
OneOf($o_1, o_2, \dots$)	$\{o_1\} \sqcup \{o_2\}$	$(\{o_1\} \sqcup \{o_2\})^{\mathcal{I}}(a) = 1$ if $a \in \{o_1^{\mathcal{I}}, o_2^{\mathcal{I}}\}$
		$(\{o_1\} \sqcup \{o_2\})^{\mathcal{I}}(a) = 0$ otherwise
restriction(R someValuesFrom(C))	$\exists R.C$	$(\exists R.C)^{\mathcal{I}}(a) = \sup_{b \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b))$
restriction(R allValuesFrom(C))	$\forall R.C$	$(\forall R.C)^{\mathcal{I}}(a) = \inf_{b \in \Delta^{\mathcal{I}}} \mathcal{J}(R^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b))$
restriction(R hasValue(o))	$\exists R.\{o\}$	$(\exists R.\{o\})^{\mathcal{I}}(a) = \sup_{b \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(a,b), \{o\}^{\mathcal{I}}(b))$
restriction(R minCardinality(m))	$\geq pR$	$(\geq pR)^{\mathcal{I}}(a) = \sup_{b_1,\dots,b_p \in \Delta^{\mathcal{I}}} t(t_{i=1}^{p} R^{\mathcal{I}}(a,b_i), t_{i<j}\{b_i \neq b_j\})$
restriction(R maxCardinality(m))	$\leq pR$	$(\leq pR)^{\mathcal{I}}(a) = \inf_{b_1,\dots,b_{p+1} \in \Delta^{\mathcal{I}}} \mathcal{J}(t_{i=1}^{p+1} R^{\mathcal{I}}(a,b_i), u_{i<j}\{b_i = b_j\})$

$(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where the domain $\Delta^{\mathcal{I}}$ is a non-empty set of objects and $\cdot^{\mathcal{I}}$ is a *fuzzy interpretation function*, which maps an individual name $a \in \mathbf{I}$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, a f-OWL class A to a membership function $\mathsf{A}^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$ and a f-OWL object property $R \in \mathbf{R}$ to a membership function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$.

An f-OWL interpretation can be extended to give semantics to fuzzy class descriptions and fuzzy class and fuzzy property axioms. The abstract syntax, the respective fuzzy DL syntax and the semantics of f-OWL class descriptions are depicted in Table 4.1. The abstract syntax, f-DL syntax and semantics of f-OWL class and property axioms are depicted in Table 4.2. Observe that specifying a membership degree along with an inequality in an individual axiom is optional. This ability will be further explained in the next section. A *fuzzy ontology*, O, is a set of f-OWL axioms. We say that a fuzzy interpretation $\mathcal{I}$ is a model of O iff it satisfies all axioms in O. A fuzzy ontology O_1 *entails* a fuzzy ontology O_2, written $O_1 \models O_2$ if every model of O_1 is a model of O_2.

There are some remarks regarding Table 4.2. Firstly, the semantics of domain axioms result from the inequations, $\sup_{b \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(a,b), 1) \leq C_i^{\mathcal{I}}(a)$, which provide the semantics for the subsumption axioms $\exists R.\top \sqsubseteq C_i$. Since the inequality holds for the supremum of the left-hand side it would hold for any $b \in \Delta^{\mathcal{I}}$ so, $t(R^{\mathcal{I}}(a,b), 1) \leq C_i^{\mathcal{I}}(a)$ and due to the boundary condition of t-norms we get the simplified inequality of Table 4.2. Similarly, the semantics of fuzzy range restrictions result by considering an arbitrary $b \in \Delta^{\mathcal{I}}$ for the inequation $1 \leq \inf_{b \in \Delta^{\mathcal{I}}} \mathcal{J}(R^{\mathcal{I}}(a,b), C_i^{\mathcal{I}}(b))$. Finally, observe that the semantics of disjoint classes are based on the GCI $C \sqcap D \sqsubseteq \bot$, rather than the axiom $C \sqsubseteq \neg D$, which is usually considered in the crisp case. While in crisp DLs the semantics of these two syntactic forms coincide, this is not *always* true in fuzzy DLs. We choose the first form since in many cases the second one provides us with some counterintuitive properties. For example, suppose we assert that a class C is disjoint from itself. Then using the first semantics and the $\min$-norm for performing fuzzy intersection we will get that $\forall a \in$

Table 4.2: Fuzzy OWL Axioms

Abstract Syntax	DL Syntax	Semantics
(Class A partial $C_1 \ \dots \ C_n$)	$A \sqsubseteq C_1 \sqcap \dots \sqcap C_n$	$A^{\mathcal{I}}(a) \leq t(C_1^{\mathcal{I}}(a), \dots, C_n^{\mathcal{I}}(a))$
(Class A complete $C_1 \ \dots \ C_n$)	$A \equiv C_1 \sqcap \dots \sqcap C_n$	$A^{\mathcal{I}}(a) = t(C_1^{\mathcal{I}}(a), \dots, C_n^{\mathcal{I}}(a))$
(EnumeratedClass A $o_1 \ \dots \ o_n$)	$A \equiv o_1 \sqcup \dots \sqcup o_n$	$A^{\mathcal{I}}(a) = 1$ if $a \in \{o_1^I, \dots, o_n^I\}$, $A^{\mathcal{I}}(a)$=0 otherwise
(SubClassOf C_1, C_2)	$C_1 \sqsubseteq C_2$	$C_1^{\mathcal{I}}(a) \leq C_2^{\mathcal{I}}(a)$
(EquivalentClasses $C_1 \ \dots \ C_n$)	$C_1 \equiv \dots \equiv C_n$	$C_1^{\mathcal{I}}(a) = \dots = C_n^{\mathcal{I}}(a)$
(DisjointClasses $C_1 \ \dots \ C_n$)	$C_i \sqcap C_j \sqsubseteq \bot$	$t(C_i^{\mathcal{I}}(a), C_j^{\mathcal{I}}(a)) = 0, 1 \leq i < j \leq n$
(SubPropertyOf R_1, R_2)	$R_1 \sqsubseteq R_2$	$R_1^{\mathcal{I}}(a,b) \leq R_2^{\mathcal{I}}(a,b)$
(EquivalentProperties $R_1 \ \dots \ R_n$)	$R_1 \equiv \dots \equiv R_n$	$R_1^{\mathcal{I}}(a,b) = \dots = R_n^{\mathcal{I}}(a,b)$
ObjectProperty(R super(R_1) ... super(R_n)	$R \sqsubseteq R_i$	$R^{\mathcal{I}}(a,b) \leq R_i^{\mathcal{I}}(a,b)$
domain(C_1) ... domain(C_k)	$\exists R.\top \sqsubseteq C_i$	$R^{\mathcal{I}}(a,b) \leq C_i^{\mathcal{I}}(a)$
range(C_1) ... range(C_h)	$\top \sqsubseteq \forall R.C_i$	$1 \leq \mathcal{J}(R^{\mathcal{I}}(a,b), C_i^{\mathcal{I}}(b))$
[InverseOf(S)]	$R \equiv S^-$	$R^{\mathcal{I}}(a,b) = (S^-)^{\mathcal{I}}(a,b)$
[Symmetric]	$R \equiv R^-$	$R^{\mathcal{I}}(a,b) = (R^-)^{\mathcal{I}}(a,b)$
[Functional]	$\top \sqsubseteq \leqslant 1R$	$\inf_{b_1,b_2 \in \Delta^{\mathcal{I}}} \mathcal{J}(t(R^{\mathcal{I}}(a,b_1), R^{\mathcal{I}}(a,b_2)), u(b_1 = b_2)) \geq 1$
[InverseFunctional]	$\top \sqsubseteq \leqslant 1R^-$	$\inf_{b_1,b_2 \in \Delta^{\mathcal{I}}} \mathcal{J}(t((R^-)^{\mathcal{I}}(a,b_1), (R^-)^{\mathcal{I}}(a,b_2)), u(b_1 = b_2)) \geq 1$
[Transitive])	$\mathsf{Trans}(R)$	$\sup_{b \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(a,b), R^{\mathcal{I}}(b,c)) \leq R^{\mathcal{I}}(a,c)$
Individual(o type(C_1) [$\bowtie$ degree(m_1)] ... type(C_n) [$\bowtie$ degree(m_n)]	$o : C_i \bowtie m_i$	$C_i^{\mathcal{I}}(o^{\mathcal{I}}) \bowtie m_i, m_i \in [0,1], 1 \leq i \leq n$
value(R_1, o_1) [$\bowtie$ degree(k_1)] ... value(R_ℓ, o_ℓ)) [$\bowtie$ degree(k_ℓ)]	$(o, o_i) : R_i \bowtie k_i$	$R_i^{\mathcal{I}}(o^{\mathcal{I}}, o_i^{\mathcal{I}}) \bowtie k_i, k_i \in [0,1], 1 \leq i \leq \ell$
Sameindividual($o_1 \ \dots \ o_n$)	$o_1 = \dots = o_n$	$o_1^{\mathcal{I}} = \dots = o_n^{\mathcal{I}}$
DifferentIndividuals($o_1 \ \dots \ o_n$)	$o_i \neq o_j$	$o_i^{\mathcal{I}} \neq o_j^{\mathcal{I}}, 1 \leq i < j \leq n$

$\Delta^{\mathcal{I}}.\min(C^{\mathcal{I}}(a), C^{\mathcal{I}}(a)) = 0 \Rightarrow \forall a \in \Delta^{\mathcal{I}}.C^{\mathcal{I}}(a) = 0$. In other words, $C \equiv \bot$, which is intuitive. On the other hand by using the second axiom we will get, $\forall a \in \Delta^{\mathcal{I}}.C^{\mathcal{I}}(a) \leq 1 - C^{\mathcal{I}}(a) \Rightarrow C^{\mathcal{I}}(a) \leq \frac{1}{2}$.

4.2 Abstract and Concrete RDF Syntax

In order to provide a complete presentation of the fuzzy OWL language in the current section we provide the abstract and concrete RDF syntax of f-OWL. We also give a number of examples to make the syntax more clear.

Table 4.3 presents the abstract syntax of fuzzy facts. We see that the usual definition of OWL facts is extended with a new element called *membership*. This element is defined by two sub-elements that of *ineqType* and *degree*. The element ineqType is used to specify the inequality type that is taking place in the instance relation. Thus, the values of this element are the inequalities, $\geq$, $\leq$, $>$ and $<$ and the equality $=$. Observe that f-OWL allows one to use the equality symbol ($=$) in order to specify an exact equality in an instance relation. This is in order to avoid writing two instance relations using the inequalities $>=$ and $<=$, as was noted in chapter 3. Finally, element degree is used to specify the membership degree that the specified instance relation holds, taken from the interval $[0, 1]$.

As we can see the membership element is optional, i.e. the user might not specify a membership for an instance relation. In that case it is reasonable to consider by default that the inequality type is of the form $=$ and the membership degree is equal to 1. More-

Table 4.3: Abstract Syntax of f-OWL

individual	::=	'Individual(' [individualID] {annotation} {'type(' type ')' [membership]} {value [membership] } ')'
membership	::=	ineqType degree
ineqType	::=	'=' | '>=' | '>' | '<=' | '<'
degree	::=	'degree(' real-number-between-0-and-1-inclusive ')'

over, we see that the membership element is placed both after the type element as well as after the value element. In the former case we can specify the membership of fuzzy facts involving an individual and a concept, while in the latter we can specify the membership between a pair of individuals and a fuzzy role.

Now let us turn our attention to the RDF/XML syntax of f-OWL. In order to demonstrate the RDF/XML syntax of f-OWL we are going to use as an example a holiday organization task. In this example we have that `Rome` is closeTo `Athens` to a degree equal to 0.75, that `Rome` is a HotPlace to a degree greater or equal than 0.7, that `Athens` is a HotPlace to a degree greater or equal than 0.8 and that `Athens` is closeTo `Cyprus` to a degree greater or equal than 0.6. As is the case with the classical OWL and RDF languages there are two different forms in which such information can be encoded. First we can use the abbreviating syntax of RDF/XML for specifying instances relations. In that case, the RDF/XML syntax is the following,

```
<HotPlace rdf:about="Rome" owlx:ineqType="≥" owlx:degree="0.7">
  <closeTo rdf:resource="Athens" owlx:degree="0.75"/>
</HotPlace>
```

On the other hand we could also use the RDF element `rdf:Description` to provide a different form of RDF/XML syntax. By using this element in the case of `Athens` we can provide the following syntax,

```
<rdf:Description rdf:about="Athens">
  <rdf:type rdf:resource="HotPlace" owlx:ineqType="≥" owlx:degree="0.8"/>
  <closeTo rdf:resource="Cyprus" owlx:ineqType="≥" owlx:degree="0.6"/>
</rdf:Description>
```

4.3 Reduction to fuzzy OWL Satisfiability

In [HPS04] a translation from OWL axioms and facts to DL axioms and assertions was provided. The aim of this translation is to reduce the problem of the entailment of OWL ontologies to the problem of satisfiability of a DL knowledge base for which implemented algorithms are known. In this section we will study the reduction in the case of fuzzy OWL and fuzzy DLs.

The first step in such reduction is to translate an f-OWL ontology to an f-DL knowledge base. More precisely we have to translate f-OWL class and property descriptions and axioms to f-DL concept and role axioms and descriptions. Such a translation is described in [HPS04]. Since we impose no syntax changes in class and property axioms and descriptions the same reduction also applies to the f-OWL language. Additionally, in the current paper we have provided the DL counterpart of the OWL class and property axioms and descriptions, (see Tables 4.1 and 4.2). The only difference with respect to the mapping in [HPS04] is the reduction of disjoint classes. Since we consider the semantics that result from the syntax $C \sqcap D \sqsubseteq \bot$ such axioms are reduced to this DL concept inclusion axiom, rather than $C \sqsubseteq \neg D$, which is used in [HPS04].

Table 4.4: From f-OWL facts to f-DL fuzzy assertions

f-OWL fragment F	Translation $\mathcal{F}(F)$
Individual($x_1 \bowtie n_1 ... x_p \bowtie n_p$)	$\mathcal{F}(a : x_1 \bowtie n_1), ..., \mathcal{F}(a : x_n \bowtie n_p)$ where a is new
$a : type(C) \bowtie n$	$a : \mathcal{V}(C) \bowtie n$
$a : type(C)$	$a : \mathcal{V}(C) = 1$
$a : value(R\ x) \bowtie n$	$\langle a, b\rangle : R \bowtie n$, $\mathcal{F}(b : x)$ where b is new
$a : value(R\ x)$	$\langle a, b\rangle : R = 1$, $\mathcal{F}(b : x)$ where b is new
$a : o$	$a = o$
Sameindividual($o_1\ \ldots\ o_n$)	$\mathcal{V}(o_i) = \mathcal{V}(o_j)\ 1 \leq i < j \leq n$
DifferentIndividuals($o_1\ \ldots\ o_n$)	$\mathcal{V}(o_i) \neq \mathcal{V}(o_j)\ 1 \leq i < j \leq n$

The most complex part of the translation, identified in [HPS04], is the translation of individual axioms (facts), because they can be stated with respect to anonymous individuals. In [HPS04] two translations were provided, one for OWL DL and one for OWL Lite. This is because the translation of OWL DL uses nominals which OWL Lite does not support. Close inspection of the abstract syntax of fuzzy individual axioms from Table 4.2, and the translations in [HPS04], reveals that the OWL Lite reduction better serves our needs in the fuzzy case even when we consider the reduction of OWL DL facts. This is due to the presence of inequality types and membership degrees. Table 4.4 defines a mapping ($\mathcal{F}$) that transforms OWL facts to f-$\mathcal{SHOIN}$ assertions. The mapping $\mathcal{V}$ represents the translation from f-OWL classes and individuals to f-DL concepts and individuals.

As a second step of the reduction we have to reduce f-$\mathcal{SHOIN}$ knowledge base entailment to f-$\mathcal{SHOIN}$ unsatisfiability. More precisely, we have to define a translation $\mathcal{G}$ such that $K \models A$ iff $K \cup \{\mathcal{G}(A)\}$ is unsatisfiable, where A represents an axiom. The definition of $\mathcal{G}$ is depicted in Table 4.5. There are some remarks regarding the definition of $\mathcal{G}$. Observe that in the reduction of the entailment of concept and role subsumption and transitive role axioms we have to check for the satisfiability of the ABoxes for all degrees

Table 4.5: From entailment to unsatisfiability

Axiom A	Transformation $\mathcal{G}(A)$
$C \sqsubseteq D$	$\{(x:C) \geq n, (x:D) < n\rangle\}$, $\forall n \in [0,1]$
$\mathsf{Trans}(R)$	$\{(x:\exists R.(\exists R.\{y\})) \geq n, (x:\exists R.\{y\}) < n\}$, $\forall n \in [0,1]$
$R \sqsubseteq S$	$\{(x:\exists R.\{y\}) \geq n, (x:\exists S.\{y\}) < n\}$, $\forall n \in [0,1]$
$(a:C)\bowtie n$	$(a:C)\neg\bowtie n$
$(\langle a,b\rangle : R)\bowtie n$	$(\langle a,b\rangle : R)\neg\bowtie n$
$a = b$	$a \neq b$
$a \neq b$	$a = b$

$n \in [0,1]$. Obsiously, such a task is practicaly impossible. Straccia [Str01] proves that for concept subsumption it suffices to check the unsatisfiability of the ABox for only two arbitrarily selected data values, n_1, n_2 from the intervals, $n_1 \in (0, 0.5]$ and $n_2 \in (0.5, 1]$. This result can be easily extended to the cases of role subsumption and transitive role axioms At last observe that the entailment of role assertions $(\langle a,b\rangle : R)\bowtie n$ is much easier in the fuzzy case, than in the crisp case. While in classical DLs the negation of roles is required, necesitating special transformation techniques [HPS04], in fuzzy DLs only a negation of the form of inequality is needed, which can easily be handled.

Regarding the reduction of f-$\mathcal{SHIF}$ entailment to f-$\mathcal{SHIF}$ satisfiability a number of issues have to be taken under consideration. More precisely, the f-$\mathcal{SHIF}$ language does not support nominals, thus the reduction method for role subsumption and transitive role axioms presented in Table 4.5, cannot be used. This is also true in the case of crisp OWL Lite [HPS04]. For that purpose a new transformation method has to be devised. Based on the translation method presented in [HPS04] we can replace each nominal concept in Table 4.5 with a new atomic concept B not present in the KB. The new mapping that uses this new notion will be denoted by $\mathcal{G}'$. This yields a translation method for f-$\mathcal{SHIF}$. Please note that, the approach taken in [HPS04] for the reduction of $\mathcal{SHIF}$ entailment to $\mathcal{SHIF}$ satisfiability is different from the approach taken here.

The reduction of OWL entailment to f-$\mathcal{SHOIN}$ satisfiability, presented in this section, together with the recent results on reasoning with expressive DLs [SST$^+$05a] with general inclusion axioms [SSSP06] and the extensions provided in the previous Chapter, implies that at the current moment we can fully support reasoning for the fuzzy OWL language f$_{KD}$-OWL. The extension of the tableaux algorithms to cover other norm operations in expressive DL languages and thus cover reasoning in OWL DL and OWL Lite or to any expressive sub-language of them is an interesting open issue.

Chapter 5

Fuzzy Rules

In the current chapter we will report on some recent results about extending Semantic Web rule languages with fuzzy set theory. More precisely, we will present a fuzzy extension to the languages SWRL and RuleML. A first account to fuzzy SWRL was presented in Deliverable 2.5.1 [PFT$^+$04], but here we revise these results by using the presentation given in [PSS$^+$06]. Finally, our presentation of the uncertainty extensions to RuleML follows the one in [SSTP05].

5.1 Fuzzy SWRL

Fuzzy rules are of the form antecedent $\rightarrow$ consequent, where atoms in both the antecedent and consequent can have weights (importance factors), which are numbers between 0 and 1. More specifically, atoms can be of the forms C(x)$*w$, P(x,y)$*w$, Q(x,z)$*w$, sameAs(x,y)$*w$, differentFrom(x,y)$*w$ or builtIn(pred,$z_1, \ldots, z_n$), where $w \in [0,1]$ is the weight of an atom, and omitting a weight is equivalent to specifying a value of 1. For instance, the following fuzzy rule axiom, inspired from the field of emotional analysis, asserts that if a man has his eyebrows raised enough and his mouth open then he is happy, and that the condition that he has his eyebrows raised is a bit more important than the condition that he has his mouth open.

$$\mathsf{EyebrowsRaised}(?a) * 0.9 \wedge \mathsf{MouthOpen}(?a) * 0.8 \rightarrow \mathsf{Happy}(?a), \quad (5.1)$$

In this example, EyebrowsRaised, MouthOpen and Happy are class URIrefs, ?a is an *individual-valued* variable, and 0.9 and 0.8 are the weights of the atoms Eyebrows-Raised(?a) and MouthOpen(?a), respectively.

In this presentation of fuzzy rules, we only consider *atomic* fuzzy rules, i.e., rules with only one atom in the consequent. The weight of an atom in a consequent, therefore, can be seen as indicating the weight that is given to the rule axiom in determining the degree with which the consequent holds. Consider, for example, the following two fuzzy rules:

$$\mathsf{parent}(?x, ?p) \wedge \mathsf{Happy}(?p) \rightarrow \mathsf{Happy}(?x) * 0.8 \quad (5.2)$$

$$\mathsf{brother}(?x, ?b) \wedge \mathsf{Happy}(?b) \rightarrow \mathsf{Happy}(?x) * 0.4, \quad (5.3)$$

which share $\mathsf{Happy}(?x)$ in the consequent. Since $0.8 > 0.4$, more weight is given to rule (5.2) than to rule (5.3) when determining the degree to which an individual is Happy.

In what follows, we formally introduce the syntax and model-theoretic semantics of fuzzy SWRL.

5.1.1 Syntax

In this section, we present the syntax of fuzzy SWRL. To make the presentation simple and clear, we use DL syntax (see the following definition) instead of the XML, RDF or abstract syntax of SWRL.

Definition 6 *Let* `a,b` *be individual URIrefs,* l *a OWL data literal,* C, D *OWL class descriptions,* r, r_1 *OWL* individual-valued *property descriptions,* r_1, r_2 *individual-valued property URIrefs,* s, s_1 data-valued *property URIrefs,* $pred$ *a datatype predicate,* $w, w_1, \ldots, w_n \in [0,1]$*,* $\vec{v}, \vec{v}_1, \ldots, \vec{v}_n$ *are (unary or binary) tuples of variables and/or individual URIrefs,* $a_1(\vec{v_1}), \ldots, a_n(\vec{v_n})$ *and* $c(\vec{v})$ *are of the forms* $C(x)$*,* $r(x,y)$*,* $s(x,z)$*,* $sameAs(x,y)$*,* $differentFrom(x,y)$*,* $\overline{m}$ *or builtIn(*$pred, z_1, \ldots, z_n$*), where* x, y *are* individual-valued *variables or individual URIrefs,* $\overline{m}$ *is a truth constant, which is a rational number between 0 and 1, and z,* $z_1, \ldots, z_n$ *are* data-valued *variables or OWL data literals.*

An f-SWRL ontology can have the following kinds of axioms:

- *class axioms:* $C \sqsubseteq D$ *(class inclusion axioms);*
- *property axioms:* $r \sqsubseteq r_1$ *(*individual-valued *property inclusion axioms),* $\mathsf{Func}(r_1)$ *(functional* individual-valued *property axioms),* $\mathsf{Trans}(r_2)$ *(transitive property axioms),* $s \sqsubseteq s_1$ *(*data-valued *property inclusion axioms),* $\mathsf{Func}(s_1)$ *(functional* data-valued *property axioms);*
- *individual axioms (facts):* $(\mathtt{a} : C) \geq m$*,* $(\mathtt{a} : C) \leq m$ *(fuzzy class assertions),* $(\langle \mathtt{a}, \mathtt{b} \rangle : r) \geq m$*,* $(\langle \mathtt{a}, \mathtt{b} \rangle : r) \leq m$ *(fuzzy* individual-valued *property assertions),* $(\langle \mathtt{a}, \mathtt{l} \rangle : r) \geq m$*,* $(\langle \mathtt{a}, \mathtt{l} \rangle : r) \leq m$ *(fuzzy* data-valued *property assertions),* $\mathtt{a} = \mathtt{b}$ *(individual equality axioms) and* $\mathtt{a} \neq \mathtt{b}$ *(individual inequality axioms);*
- *rule axioms:* $a_1(\vec{v_1}) * w_1 \wedge \cdots \wedge a_n(\vec{v_n}) * w_n \rightarrow c(\vec{v}) * w$ *(fuzzy rule axioms).*

Omitting a degree or a weight is equivalent to specifying the value of 1. ◇

According to the above definition, f-SWRL extends SWRL with fuzzy class assertions, fuzzy property assertions and fuzzy rule axioms. We have some remarks here. Firstly, in f-SWRL, there are two (i.e. $\geq$ and $\leq$) kinds of fuzzy assertions; as we have already mentioned, we can simulate the form of $(a : C) = m$ by considering two assertions of the form $(a : C) \geq m$ and $(a : C) \leq m$. Secondly, although f-SWRL supports degrees in fuzzy assertions, it does not support degrees in fuzzy class axioms and fuzzy property axioms because it is not very clear how to obtain degrees for them. Nevertheless, it is worth noting that fuzzy class axioms and fuzzy property axioms *have* fuzzy interpretations instead of crisp interpretations (see Section 5.1.3). Furthermore, we allow the use of truth constants $\overline{m}$ [Pav79, Haj98] in the consequence of a fuzzy rule axiom. This could enable us to simulate fuzzy assertions of the form $(a : C) \leq m$ with fuzzy rule axioms (see Section 5.1.3).

5.1.2 Constraints on Semantics

In order to make the semantics of f-SWRL more intuitive, in this section we briefly clarify the constraints of our desired semantics for f-SWRL. The proposed constraints provide a unified framework for giving model theoretic semantics for f-SWRL based on fuzzy intersections (t-norms), fuzzy union (s-norms), fuzzy negations, fuzzy implications and weight operations $g(w, d) : [0, 1]^2 \to [0, 1]$, i.e. how to handle the degree d of an atom (in antecedents) and its weight w. The properties of weight operations are defined later on this section.

Firstly, one of the most useful relationships which is used to manipulate expressions in propositional logic is the *modus ponen*, which states that $A \cap (A \Rightarrow B) \Rightarrow B$ (if A is true and A implies B, then B is also true). This suggests the following constraint on fuzzy implications.

Constraint 7 *The fuzzy implications used in the semantics of f-SWRL should satisfy the modus ponen:*

$$\mathcal{J}(t(a, \mathcal{J}(a, b)), b) = 1.$$

It is easy to verify that, e.g., the following two sets of fuzzy operations satisfy the above constraint:

- $\{t(a, b) = \min(a, b), \mathcal{J}_R(a, b) = \sup\{x \in [0, 1] \mid t(a, x) \leq b\}\}$,
- $\{t(a, b) = a \cdot b, \mathcal{J}_R(a, b) = \sup\{x \in [0, 1] \mid t(a, x) \leq b\}\}$,

while the set of fuzzy operations $\{t(a, b) = \min(a, b), u(a, b) = \max(a, b), c(a) = 1 - a, \mathcal{J}_S(a, b) = u(c(a), b)\}$ does not (e.g., when $a = 0.4, b = 0.5$). In short, R-implication satisfies Constraint 7, while S-implication does not.

Secondly, we require the weight operations $g(w, d)$ in antecedents satisfy the following properties.

Constraint 8 *The weight operations $g(w, d)$ used in the semantics of f-SWRL should satisfy the following properties:*

1. *monotone in d: if $d_1 < d_2$ then $g(w, d_1) < g(w, d_2)$,*
2. *$g(0, d) = 1, g(1, d) = d$.*

The intuition of Property 1 is immediate. Property 2 ensures that the weight 0 would not affect the result of fuzzy intersections in the antecedent, and that the full membership degree would participate in fuzzy intersections when the weight is 1.

It is easy to verify that, e.g., the following two weight operations satisfy the above constraint:

- $g(a, b) = \begin{cases} a \cdot b & \text{if } a \neq 0 \\ 1 & \text{if } a = 0 \end{cases}$,
- $g(a, b) = \mathcal{J}_R(a, b)$,

while the weight operation $g(a, b) = \min(a, b)$ does not (e.g. when $a = 0$). Observe that the former type of weight operation is t-norm based, while the latter one is R-implication based.

Thirdly, in order to enable the use of weights in the head atoms as the weights of the rule axiom, we have the following constraint.

Constraint 9 *Given a fuzzy rule $A \to c*w$, where A is the antecedent of the rule and c is the consequent atom with weight w, the semantics of f-SWRL should satisfy the following property:*

$$\mathcal{J}(A(\mathcal{I}), c(\mathcal{I})) \geq w,$$

where $A(\mathcal{I})$ and $c(\mathcal{I})$ are interpretations of A and c, respectively.

Intuitively speaking, the above constraint requires that the degree of fuzzy implication should be no less than the weight. This constraint is inspired by Theorem 5 from [DP01], which shows an important property of the weighted rules of the form $A \xrightarrow{\theta} C$, where θ is a weight of the rule.

Furthermore, individual axioms (or facts) are special forms of rule axioms in SWRL. This suggests yet another constraint on the semantics of f-SWRL.

Constraint 10 *The semantics of f-SWRL should ensure that fuzzy individual axioms (fuzzy facts) are special forms of fuzzy rule axioms.*

It is worth noting that we do not require fuzzy class (or property) axioms be special forms of fuzzy rule axioms. In some decidable sub-languages of SWRL, such as the DL-safe SWRL [MSS04], class (or property) axioms are not special forms of rule axioms.

5.1.3 Model-theoretic Semantics

In this section, we give a model-theoretic semantics for fuzzy SWRL, based on the constraints specified in the previous section. Although many f-SWRL axioms share the same syntax as their counterparts in SWRL, such as class inclusion axioms, they have different semantics because we use fuzzy interpretations in the model-theoretic semantics of f-SWRL.

Before we provide a model-theoretic semantics for f-SWRL, we introduce the notions of datatype predicates and datatype predicate maps.

Definition 11 (Datatype Predicate) *A* datatype predicate *(or simply* predicate*) p is characterised by an arity $a(p)$, or a minimum arity $a_{min}(p)$ if p can have multiple arities, and a predicate extension (or simply* extension*) $E(p)$.* ◇

For example, $=^{int}$ is a datatype predicate with arity $a(=^{int}) = 2$ and extension $E(=^{int}) = \{\langle i_1, i_2\rangle \in V(integer)^2 \mid i_1 = i_2\}$, where $V(integer)$ is the set of all integers. Datatypes can be regarded as *special* predicates with arity 1 and predicate extensions equal to their value spaces; e.g., the datatype $integer$ can be seen as a predicate with arity $a(integer) = 1$ and predicate extension $E(integer) = V(integer)$.[1]

Definition 12 (Predicate Map) *We consider a predicate map M_p that is a partial mapping from predicate URI references to predicates.* ◇

Intuitively, datatype predicates (resp. datatype predicate URIrefs) in M_p are called built-in datatype predicates (resp. datatype predicate URIrefs) w.r.t. M_p. Note that allowing the datatype predicate map to vary allows different implementations of f-SWRL to implement different datatype predicates.

Based on the constraints we specified in the previous section, we define the semantics of f-SWRL as follows.

Definition 13 *Let c, t, u be fuzzy negations, fuzzy intersections and fuzzy unions, g weight operations that satisfy Constraints 8. Due to Constraint 7, we choose the R-implication as the fuzzy implication. Given a datatype predicate map M_p, a fuzzy interpretation is a triple $\mathcal{I} = \langle\Delta^{\mathcal{I}}, \Delta_{\mathbf{D}}, \cdot^{\mathcal{I}}\rangle$, where the abstract domain $\Delta^{\mathcal{I}}$ is a non-empty set, the datatype domain contains at least all the data values in the extensions of built-in datatype predicates in M_p, and $\cdot^{\mathcal{I}}$ is a fuzzy interpretation function, which maps*

1. *individual URIref and* individual-valued *variables to elements of $\Delta^{\mathcal{I}}$,*
2. *a class description C to a membership function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0, 1]$,*

[1] See [Pan04] for detailed discussions on the relationship between datatypes and datatype predicates.

3. *an* individual-valued *property URIref* r *to a membership function* $r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$,

4. *an* data-valued *property URIref* q *to a membership function* $q^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}} \to [0,1]$,

5. *a truth constant* $\overline{m}$ *to itself:* $\overline{m}^{\mathcal{I}} = m$,

6. *a built-in datatype predicate URIref* $pred$ *to its extension* $pred^{\mathcal{I}} = E(\mathbf{M}_p(pred)) \in (\Delta_{\mathbf{D}})^n$*, where* $n = a(\mathbf{M}_p(pred))$*, so that*

$$builtIn^{\mathcal{I}}(pred, z_1, \ldots, z_n) = \begin{cases} 1 \text{ if } \langle z_1^{\mathcal{I}}, \ldots, z_n^{\mathcal{I}} \rangle \in pred^{\mathcal{I}} \\ 0 \text{ otherwise,} \end{cases}$$

7. *the built-in property* $sameAs$ *to a membership function*

$$sameAs^{\mathcal{I}}(x,y) = \begin{cases} 1 \text{ if } x^{\mathcal{I}} = y^{\mathcal{I}} \\ 0 \text{ otherwise,} \end{cases}$$

8. *the built-in property* $differentFrom$ *to a membership function*

$$differentFrom^{\mathcal{I}}(x,y) = \begin{cases} 1 \text{ if } x^{\mathcal{I}} \neq y^{\mathcal{I}} \\ 0 \text{ otherwise.} \end{cases}$$

The semantics for fuzzy class descriptions have been previously presented in Table 3.1.

A fuzzy interpretation $\mathcal{I}$ *satisfies a class inclusion axiom* $C \sqsubseteq D$*, written* $\mathcal{I} \models C \sqsubseteq D$*, if* $\forall o \in \Delta^{\mathcal{I}}, C^{\mathcal{I}}(o) \leq D^{\mathcal{I}}(o)$.

A fuzzy interpretation $\mathcal{I}$ *satisfies an* individual-valued *property inclusion axiom* $r \sqsubseteq r_1$*, written* $\mathcal{I} \models r \sqsubseteq r_1$*, if* $\forall o, q \in \Delta^{\mathcal{I}}, r^{\mathcal{I}}(o,q) \leq r_1^{\mathcal{I}}(o,q)$. $\mathcal{I}$ *satisfies a functional* individual-valued *property axiom* $\mathsf{Func}(r_1)$*, written* $\mathcal{I} \models \mathsf{Func}(r_1)$*, if*

$$\forall o \in \Delta^{\mathcal{I}}, \inf_{q_1, q_2 \in \Delta^{\mathcal{I}}} \mathcal{J}(t(R^{\mathcal{I}}(o,q_1), R^{\mathcal{I}}(o,q_2)), q_1 = q_2) \geq 1.$$

$\mathcal{I}$ *satisfies a transitive property axiom* $\mathsf{Trans}(r_2)$*, written* $\mathcal{I} \models \mathsf{Trans}(r_2)$*, if* $\forall o, q \in \Delta^{\mathcal{I}}, r_2^{\mathcal{I}}(o,q) = \sup_{p \in \Delta^{\mathcal{I}}} t[r_2^{\mathcal{I}}(o,p), r_2^{\mathcal{I}}(p,q)]$*, where* t *is a triangular norm. A fuzzy interpretation* $\mathcal{I}$ *satisfies a* data-valued *property inclusion axiom* $s \sqsubseteq s_1$*, written* $\mathcal{I} \models s \sqsubseteq s_1$*, if* $\forall \langle o, l \rangle \in \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}}, s^{\mathcal{I}}(o,l) \leq s_1^{\mathcal{I}}(o,l)$. $\mathcal{I}$ *satisfies a functional* data-valued *property axiom* $\mathsf{Func}(s_1)$*, written* $\mathcal{I} \models \mathsf{Func}(s_1)$*, if*

$$\forall o \in \Delta^{\mathcal{I}}, \inf_{l_1, l_2 \in \Delta^{\mathcal{I}}} \mathcal{J}(t(R^{\mathcal{I}}(o,l_1), R^{\mathcal{I}}(o,l_2)), l_1 = l_2) \geq 1.$$

A fuzzy interpretation $\mathcal{I}$ *satisfies a fuzzy class assertion* $(\mathsf{a} : C) \geq m$*, written* $\mathcal{I} \models (\mathsf{a} : C) \geq m$*, if* $C^{\mathcal{I}}(\mathsf{a}) \geq m$. $\mathcal{I}$ *satisfies a fuzzy* individual-valued *property assertion*

$(\langle \mathtt{a,b} \rangle : r) \geq m_2$, *written* $\mathcal{I} \models (\langle \mathtt{a,b} \rangle : r) \geq m_2$, *if* $r^{\mathcal{I}}(\mathtt{a,b}) \geq m_2$. $\mathcal{I}$ *satisfies a fuzzy* data-valued *property assertion* $(\langle \mathsf{a}, l \rangle : s) \geq m_3$, *written* $\mathcal{I} \models (\langle \mathsf{a}, l \rangle : s) \geq m_3$, *if* $s^{\mathcal{I}}(\mathsf{a}, l) \geq m_3$. *The semantics of fuzzy assertions using* $\leq$ *are defined analogously.* $\mathcal{I}$ *satisfies an individual equality axiom* $\mathsf{a} = \mathsf{b}$, *written* $\mathcal{I} \models \mathsf{a} = \mathsf{b}$, *if* $\mathsf{a}^{\mathcal{I}} = \mathsf{b}^{\mathcal{I}}$. $\mathcal{I}$ *satisfies an individual inequality axiom* $\mathsf{a} \neq \mathsf{b}$, *written* $\mathcal{I} \models \mathsf{a} \neq \mathsf{b}$, *if* $\mathsf{a}^{\mathcal{I}} \neq \mathsf{b}^{\mathcal{I}}$.

A fuzzy interpretation $\mathcal{I}$ *satisfies a fuzzy rule axiom* $a_1(\vec{v_1}) * w_1 \wedge \cdots \wedge a_n(\vec{v_n}) * w_n \rightarrow c(\vec{v}) * w$, *written* $\mathcal{I} \models a_1(\vec{v_1}) * w_1 \wedge \cdots \wedge a_n(\vec{v_n}) * w_n \rightarrow c(\vec{v}) * w$, *if* $t(g(w_1, a_1^{\mathcal{I}}(\vec{v_1}^{\mathcal{I}})), \ldots, g(w_n, a_n^{\mathcal{I}}(\vec{v_n}^{\mathcal{I}}))) \leq \mathcal{J}_R(w, c^{\mathcal{I}}(\vec{v}^{\mathcal{I}}))$. ◇

There are some remarks on the above definition. Firstly, as we have seen in the previous section, only R-implication satisfies Constraint 7. Therefore, we implicitly use R-implication for fuzzy rule axioms (see below). In fact, given a fuzzy rule axiom $A \rightarrow C$, Definition 13 asserts that an fuzzy interpretation $\mathcal{I}$ satisfies $A \rightarrow C$ if $A(\mathcal{I}) \leq C(\mathcal{I})$, where $A(\mathcal{I})$ and $C(\mathcal{I})$ are interpretations of the antecedent A and conclusion C, respectively. By applying using the property of R-implications introduced in Chapter 2, it follows that $\mathcal{J}_R(A(\mathcal{I}), C(\mathcal{I})) = 1$. One of the consequences of such semantics is the support of *chaining* of rules. Suppose that we have two fuzzy rule axioms $A_1 \rightarrow C_1, C_1 \rightarrow C_2$, if an fuzzy interpretation $\mathcal{I}$ satisfies both of them, i.e. $A_1(\mathcal{I}) \leq C_1(\mathcal{I})$ and $C_1(\mathcal{I}) \leq C_2(\mathcal{I})$, it follows $A_1(\mathcal{I}) \leq C_2(\mathcal{I})$. In other words, $\mathcal{I}$ also satisfies the fuzzy rule axiom $A_1 \rightarrow C_2$.

Let us conclude this section by showing that f-SWRL satisfies all the constraints presented in Section 5.1.2.

Lemma 14 *Given a f-SWRL rule axiom* $A \rightarrow c * w$, *where* A *is the antecedent of the rule and* c *is the consequent atom with weight* w, *we have* $\mathcal{J}_R(A(\mathcal{I}), c(\mathcal{I})) \geq w$, *where* $A(\mathcal{I})$ *and* $c(\mathcal{I})$ *are interpretations of* A *and* c, *respectively.*

Proof: According to the Definition 4, we have $A(\mathcal{I}) \leq \mathcal{J}_R(w, c(\mathcal{I}))$. Due to Property 1 of Lemma 1, we have $t(w, A(\mathcal{I})) \leq c(\mathcal{I})$; i.e., $t(A(\mathcal{I}), w) \leq c(\mathcal{I})$. Due to Property 1 of Lemma 1 again, we have $\mathcal{J}_R(A(\mathcal{I}), c(\mathcal{I})) \geq w$.□

Lemma 15 *In f-SWRL, fuzzy assertions are special forms of fuzzy rule axioms.*

Proof: $(a : C) \geq m$ can be simulated by $\top(a) \rightarrow C(a) * m$. According to Definition 4, we have $1 \leq \mathcal{J}_R(m, C^{\mathcal{I}}(a))$. Due to Property 2 of Lemma 1, we have $C^{\mathcal{I}}(a) \geq m$, which is the interpretation of $(a : C) \geq m$.

$(a : C) \leq m$ can be simulated by $C(a) \rightarrow \overline{m}$, where $\overline{m}$ is a truth constant. According to Definition 4, we have $C^{\mathcal{I}}(a) \leq \mathcal{J}_R(1, m)$. Due to Property 4 of Lemma 1, we have $C^{\mathcal{I}}(a) \leq m$, which is the interpretation of $(a : C) \leq m$.

Similarly, $(\langle a, b \rangle : r) \geq m$ can be simulated by $\top(a) \wedge \top(b) \rightarrow r(a, b) * m$, and $(\langle a, b \rangle : r) \leq m$ can be simulated by $r(a, b) \rightarrow \overline{m}$.□

Based on Definition 4, Lemma 2 and Lemma 3, we have the following theorem.

Theorem 16 *f-SWRL satisfies Constraints 1-4.*

5.2 Examples

In this section, we use some examples to further illustrate the semantics of f-SWRL.

Example 2 *Suppose that a casting or production company retains fuzzy knowledge, in the form of f-SWRL rules, about specific characteristics of the models it represents, to advertising companies. An excerpt of such a fuzzy knowledge base could contain the following axioms:*

- `Mary` *is* Tall *with a degree no less than 0.65:* $(\texttt{Mary} : \mathsf{Tall}) \geq 0.65$.
- `Mary` *is* Light *with a degree no less than 0.9:* $(\texttt{Mary} : \mathsf{Light}) \geq 0.9$.
- `Susan` *is* Tall *with a degree no less than 0.8:* $(\texttt{Susan} : \mathsf{Tall}) \geq 0.8$.
- `Susan` *is* Light *with a degree no less than 0.6:* $(\texttt{Susan} : \mathsf{Light}) \geq 0.6$.
- *One is* Thin *if one is* Tall *(with importance factor 0.7) and* Light *(with importance factor 0.8):*
$$\mathsf{Tall}(?p) * 0.7 \wedge \mathsf{Light}(?p) * 0.8 \rightarrow \mathsf{Thin}(?p).$$

The interpretation of the above rule axiom is as follows:

$$t(g(0.7, \mathsf{Tall}^{\mathcal{I}}(?p^{\mathcal{I}})), g(0.8, \mathsf{Light}^{\mathcal{I}}(?p^{\mathcal{I}}))) \leq \mathcal{J}_R(1, \mathsf{Thin}^{\mathcal{I}}(?p^{\mathcal{I}})).$$

In this example, we first use the following operations:

$$t(a,b) = \min(a,b), \mathcal{J}_R(a,b) = \begin{cases} 1 & \textit{if } a \leq b \\ b & \textit{if } a > b \end{cases}, g(a,b) = \begin{cases} a \cdot b & \textit{if } a \neq 0 \\ 1 & \textit{if } a = 0 \end{cases}$$

According to Definition 13, we have

$$\mathsf{Thin}^{\mathcal{I}}(\texttt{Mary}^{\mathcal{I}}) \geq \min(0.7 \cdot 0.65, 0.8 \cdot 0.9) = \min(0.455, 0.72) = 0.455,$$

while

$$\mathsf{Thin}^{\mathcal{I}}(\texttt{Susan}^{\mathcal{I}}) \geq \min(0.7 \cdot 0.8, 0.8 \cdot 0.6) = \min(0.56, 0.48) = 0.48.$$

As a result, by using t-norm based weight operations `Susan` *is thinner than* `Mary`.

If we choose another set of operations, the conclusion, however, can be completely different.

For example, now we use the following operations:

$$t(a,b) = a \cdot b, \mathcal{J}_R(a,b) = \begin{cases} 1 & \textit{if } a \le b \\ b/a & \textit{if } a > b \end{cases}, g(a,b) = \mathcal{J}_R(a,b).$$

According to Definition 13, we have $\mathsf{Thin}^{\mathcal{I}}(\texttt{Mary}^{\mathcal{I}}) \ge \mathcal{J}_R(0.7, 0.65) \cdot \mathcal{J}_R(0.8, 0.9) = 0.929 \cdot 1 = 0.929$*, while* $\mathsf{Thin}^{\mathcal{I}}(\texttt{Susan}^{\mathcal{I}}) \ge \mathcal{J}_R(0.7, 0.8) \cdot \mathcal{J}_R(0.8, 0.6) = 1 \cdot 0.75 = 0.75$*. As a result,* `Mary` *is substantially thinner than* `Susan` *in this setting.* ♢

The above example indicates that t-torm based weights give quite different meaning than $\mathcal{J}_R$ based weights.

Example 3 *Suppose we have an f-SWRL knowledge base as follows:*

- `Tom` *is* Happy *with a degree no less than 0.7:* $(\texttt{Tom} : \mathsf{Happy}) \ge 0.7$*,*
- `Tom` *is a* $parent$ *of* `Jane`*:* $\langle \texttt{Jane}, \texttt{Tom} \rangle : parent$*,*
- `Tom` *is a* $brother$ *of* `Kate`*:* $\langle \texttt{Kate}, \texttt{Tom} \rangle : brother$*,*
- *if one's* $parent$ *is* Happy*, then one is* Happy *(with importance factor 0.8):*

$$\mathsf{parent}(?x, ?p) \wedge \mathsf{Happy}(?p) \rightarrow \mathsf{Happy}(?x) * 0.8,$$

- *if one's* $brother$ *is* Happy*, then one is* Happy *(with importance factor 0.4):*

$$\mathsf{brother}(?x, ?b) \wedge \mathsf{Happy}(?b) \rightarrow \mathsf{Happy}(?x) * 0.4.$$

Let us use the two sets of operations in Example 2 with this knowledge base.

Firstly, we use the following operations:

$$t(a,b) = \min(a,b), \mathcal{J}_R(a,b) = \begin{cases} 1 & \textit{if } a \le b \\ b & \textit{if } a > b \end{cases}, g(a,b) = \begin{cases} a \cdot b & \textit{if } a \ne 0 \\ 1 & \textit{if } a = 0 \end{cases}$$

According to Definition 13, we have $\mathcal{J}_R(0.8, \mathsf{Happy}^{\mathcal{I}}(\texttt{Jane}^{\mathcal{I}})) \ge \min(1 \cdot 1, 1 \cdot 0.7) = 0.7$*. Due to Property 1 of Lemma 1, we have* $\mathsf{Happy}^{\mathcal{I}}(\texttt{Jane}^{\mathcal{I}}) \ge 0.7$*. As for* `Kate`*, we have* $\mathcal{J}_R(0.4, \mathsf{Happy}^{\mathcal{I}}(\texttt{Kate}^{\mathcal{I}})) \ge \min(1 \cdot 1, 1 \cdot 0.7) = 0.7$*; hence,* $\mathsf{Happy}^{\mathcal{I}}(\texttt{Kate}^{\mathcal{I}}) \ge 0.4$*. Hence,* `Jane` *is happier than* `Kate`*.*

Now we use the following operations:

$$t(a,b) = a \cdot b, \mathcal{J}_R(a,b) = \begin{cases} 1 & \textit{if } a \le b \\ b/a & \textit{if } a > b \end{cases}, g(a,b) = \mathcal{J}_R(a,b).$$

According to Definition 13, we have $\mathcal{J}_R(0.8, \mathsf{Happy}^{\mathcal{I}}(\texttt{Jane}^{\mathcal{I}})) \ge \mathcal{J}_R(1,1) \cdot \mathcal{J}_R(1, 0.7) = 0.7$*; hence,* $\mathsf{Happy}^{\mathcal{I}}(\texttt{Jane}^{\mathcal{I}}) \ge t(0.8, 0.7) = 0.56$*. As for* `Kate`*, we have*

$$\mathcal{J}_R(0.4, \mathsf{Happy}^{\mathcal{I}}(\texttt{Kate}^{\mathcal{I}})) \ge \mathcal{J}_R(1,1) \cdot \mathcal{J}_R(1, 0.7) = 0.7;$$

hence, $\mathsf{Happy}^{\mathcal{I}}(\texttt{Kate}^{\mathcal{I}}) \ge t(0.4, 0.7) = 0.28$*. Again,* `Jane` *is happier than* `Kate`*.* ♢

So far we have only seen fuzzy assertions of the form $(a : C) \geq m$; in the next example, we will use fuzzy assertions of the form $(a : C) \leq m$.

Example 4 *Suppose we have a slightly different f-SWRL knowledge base from that in the previous example.*

- `Jane` *is* Happy *with a degree no larger than 0.75:* $(\texttt{Jane} : \mathsf{Happy}) \leq 0.75$,
- `Kate` *is* Happy *with a degree no larger than 0.85:* $(\texttt{Kate} : \mathsf{Happy}) \leq 0.85$,
- `Tom` *is a* $parent$ *of* `Jane`*:* $\langle \texttt{Jane}, \texttt{Tom} \rangle : parent$,
- `Tom` *is a* $brother$ *of* `Kate`*:* $\langle \texttt{Kate}, \texttt{Tom} \rangle : brother$,
- *if one's* $parent$ *is* Happy*, then one is* Happy *(with importance factor 0.8):*

$$\mathsf{parent}(?x, ?p) \wedge \mathsf{Happy}(?p) \rightarrow \mathsf{Happy}(?x) * 0.8 \tag{5.4}$$

- *if one's* $brother$ *is* Happy*, then one is* Happy *(with importance factor 0.4):*

$$\mathsf{brother}(?x, ?b) \wedge \mathsf{Happy}(?b) \rightarrow \mathsf{Happy}(?x) * 0.4, \tag{5.5}$$

Here we use the following operations:

$$t(a,b) = \min(a,b), \mathcal{J}_R(a,b) = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}, g(a,b) = \begin{cases} a \cdot b & \text{if } a \neq 0 \\ 1 & \text{if } a = 0 \end{cases}.$$

From (5.4), we have $\mathsf{Happy}^{\mathcal{I}}(\texttt{Tom}^{\mathcal{I}}) \leq \mathcal{J}_R(0.8, \mathsf{Happy}^{\mathcal{I}}(\texttt{Jane}^{\mathcal{I}})) \leq \mathcal{J}_R(0.8, 0.75) = 0.75$*. Hence,* $\mathsf{Happy}^{\mathcal{I}}(\texttt{Tom}^{\mathcal{I}}) \leq 0.75$*. From (5.5), we have* $\mathcal{J}_R(0.4, \mathsf{Happy}^{\mathcal{I}}(\texttt{Kate}^{\mathcal{I}})) \geq \mathsf{Happy}^{\mathcal{I}}(\texttt{Tom}^{\mathcal{I}})$*. Then we have* $\mathsf{Happy}^{\mathcal{I}}(\texttt{Kate}^{\mathcal{I}}) \geq \min(0.4, \mathsf{Happy}^{\mathcal{I}}(\texttt{Tom}^{\mathcal{I}}))$.

It is easy to verify that we have the same results if we use the other set of operations. $\diamondsuit$

5.3 Uncertainty extensions to RuleML

RuleML [2] is a family of markup languages intended in encoding and exchanging rules in the Semantic Web. It includes languages capable of capturing many widely known systems of rule languages, such as datalog, hornlog, production rules, and more. For example, returning to our models knowledge base, we can use RuleML to encode a rule which determines which models qualify for a certain commercial and which do not. More precisely, such a rule could look as follows,

[2] `http://www.ruleml.org`

```
<Implies>
  <head>
    <Atom>
      <_opr><Rel>qualify</Rel></_opr>
      <Var>person</Var>
    </Atom>
  </head>
  <body>
    <And>
      <Atom>
        <_opr><Rel>long_hair</Rel></_opr>
        <Var>person</Var>
      <Atom>
      <Neg>
        <Atom>
          <_opr><Rel>good_quality</Rel></_opr>
          <Var>person</Var>
        <Atom>
      </Neg>
    </And>
  </body>
</Implies>
```

Intuitively, the above rule states that if ones hair are long and of good quality she qualifies for acting in the commercial.

Like SWRL, RuleML is not capable of representing vague and imprecise knowledge, hence a fuzzy extension is required to capture such expressiveness. As it is argued in other fuzzy extensions of concept based or rule based systems, the only syntactic change that is needed is in the representation of facts (assertions). In this case we have the notion of a *fuzzy assertion* [SST^{+}05b], or a *fuzzy fact*. Intuitively, a fuzzy fact asserts the minimum degree to which a specific individual belongs to a specific atom. We have seen several examples of fuzzy facts in the previous section.

We can make these fuzzy assertions explicit to the system by encoding them as RuleML fuzzy facts [SSTP05]. Some the above mentioned fuzzy facts can be encoded by the following syntax:

```
<Atom>
  <degree><Data>0.8</Data></degree>
  <_opr><Rel>long_hair</Rel></_opr>
  <Ind>Mary</Ind>
</Atom>
```

```
<Atom>
  <degree><Data>0.6</Data></degree>
  <_opr><Rel>long_hair</Rel></_opr>
  <Ind>Susan</Ind>
</Atom>
```

Moreover, as for the hair quality one could write the following fuzzy facts:

```
<Atom>
  <degree><Data>0.6</Data></degree>
  <_opr><Rel>good_quality</Rel></_opr>
  <Ind>Mary</Ind>
</Atom>

<Atom>
  <_opr><Rel>good_quality</Rel></_opr>
  <Ind>Susan</Ind>
</Atom>
```

Observe that in the last fuzzy fact, for Susan's hair quality, we have not specified the degree to which we believe that the object "Susan" belongs to the fuzzy atom "good_quality". In this case the membership degree of the tuple to the fuzzy atom is taken to be 1 by default.

From the above examples we can see that the syntactic changes that need to take place are minimal and only involve the syntax of fuzzy facts. So the additional change that needs to take place in the XML Schema definition of the element `Atom` [HBG+05] is the following:

```
<xs:group name="Atom.content">
...................
  <xs:choice>
    <xs:sequence>
      <xs:element name="degree" type="degree.type" minOccurs="0"
maxOccurs="1"/>
    <xs:choice>
      <xs:element ref="opr"/>
....................
</xs:group>
```

The XML Schema definition of the new tag `degree` could be given by the following schema definition:

```
<xs:attributeGroup name="degree.attlist"/>
  <xs:group name="degree.content">
    <xs:sequence>
      <xs:element ref="Data"/>
    </xs:sequence>
  </xs:group>
  <xs:complexType name="degree.type">
    <xs:group ref="degree.content"/>
    <xs:attributeGroup ref="degree.attlist"/>
  </xs:complexType>
<xs:element name="degree" type="degree.type"/>
```

Chapter 6

Conclusions

In this report we have investigated fuzzy extensions to Semantic Web languages, like the ontology languages of Description Logics and OWL, and the rule languages SWRL and RuleML. Such extensions have gained considerable attention within the research community since many applications from different domains are facing a vast amount of uncertain and vague knowledge and information. Hence, in these domains applications have to directly deal with such types of information in order to provide good quality results. Examples of such domains are multimedia analysis, where the color, shape and texture of objects within an image are vague and geospatial applications, where spatial relations relative to distances, like "close", "far", etc are also imprecise.

In the current deliverable we report on very important theoretical results in this direction, achieved as work of the current work package. More precisely, as we have presented, we can now fully support reasoning in a fuzzy OWL language for some specific fuzzy operations. This implies an extension of the syntax and semantics of the OWL language, the reduction of fuzzy OWL ontologies to fuzzy Description Logics, the syntax and semantics of fuzzy Description Logics and a reasoning algorithm for the very expressive fuzzy DL, f_{KD}-$\mathcal{SHOIN}$. Finally, following the current research in the Semantic Web, we investigate the extension of several popular rule languages, like the SWRL language and the RuleML.

In addition to the above theoretical results we have also started the implementation of a prototype reasoning platform for fuzzy Description Logics. This new platform, FiRE, currently supports reasoning in the language f_{KD}-$\mathcal{SHIN}$ with simple TBoxes and with no optimization.

As for future research we are mainly focusing in three different directions. Firstly, we are investigating reasoning support in expressive fuzzy DLs for other norm operations, as well as reasoning in various clusters of combinations of fuzzy DLs and fuzzy rules. Secondly, we are seeking ways fuzzy DLs and OWL can assist the process of image and video analysis, identification and automatic annotation, hence providing knowledge based image analysis applications. Finally, we focus on implementations. More precisely, we are investigating the application of DL optimization techniques to the case of fuzzy DL algorithms, extending FiRE engine to support nominals and GCIs and integrating FiRE with RDF and OWL stores to support large number of data.

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